FUN WITH DIRAC EIGENVALUES

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Eigenvalues popular for discussion

- chiral condensate and density of small eigenvalues
  - Banks-Casher formula
- approximations to the Ginsparg Wilson relation
  - eigenvalues near "circles"
- projection issues for the overlap/domain wall operators
  - undefined when $D_W^\dagger D_W$ not invertible
  - need a gap in the Wilson operator spectrum

Dangers

- eigenvalues depend on gauge fields
- gauge fields depend on eigenvalues
- Highly non-linear system!

Generic path integral

$$Z = \int (dA)(d\psi)(d\overline{\psi}) \; e^{-S_G(A)+\overline{\psi}D(A)\psi}$$

Integrate out fermions

$$Z = \int (dA) \, |D(A)| \, e^{-S_G(A)}$$

Determinant is product of eigenvalues

$$D(A)\psi_i = \lambda_i \psi_i$$

$$Z = \int (dA) \, e^{-S_G(A)} \prod_i \lambda_i$$

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Eigenvalue "density"

$$\rho(x+iy) = \frac{1}{VZ} \int (dA) \, |D(A)| \, e^{-S_G(A)} \sum_i \delta(x - \mathrm{Re}\lambda_i(A)) \delta(y - \mathrm{Im}\lambda_i(A))$$

- V dimension of D; proportional to system volume

- May not be positive if $|D|$ is not

- Hermiticity condition (no chemical potential)

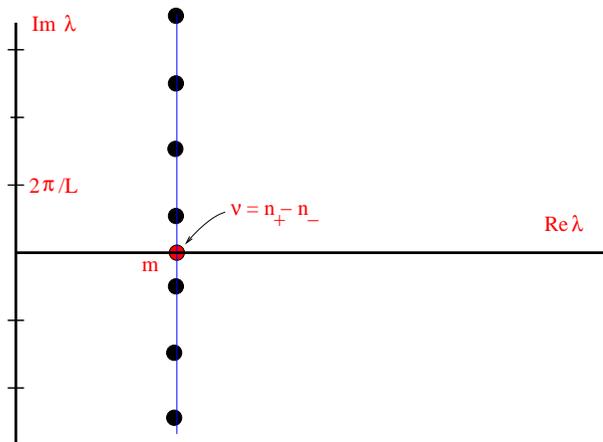 - $\gamma_5 D \gamma_5 = D^\dagger$

  - $\rho(z) = \rho(z^*)$

Repeat warning:

- λ depends on A which is weighted by λ which depends on $A \ldots$

Continuum

$$D = \gamma_\mu(\partial_\mu + igA_\mu) + m$$

$$\rho(x + iy) = \delta(x - m)\tilde{\rho}(y)$$
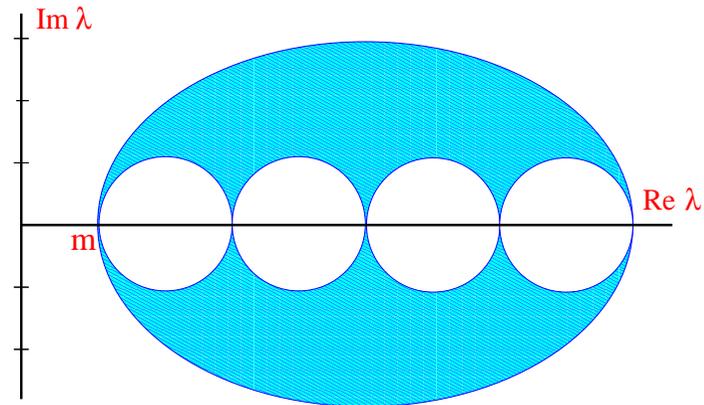

Banks and Casher, multiple flavors, vanishing mass

- $\langle\overline{\psi}\psi\rangle \neq 0$ correlates with $\tilde{\rho}(0) \neq 0$

Index theorem: consider eigenmodes with real eigenvalues

- γ_5 commutes with D when restricted to this set
- chirality ± 1
- winding number $\nu = n_+ - n_-$
- matches winding from smooth gauge field topology

Lattice

- free Wilson fermions

- doublers given large real part

Turn on gauge fields

- D no longer normal, i.e. $[D, D^\dagger] \neq 0$

- eigenvalues spread out, remain in complex conjugate pairs

- some eigenvalue pairs collide and become real

  - continuous spectrum of eigenvalues along real axis

  - index theorem for smooth fields

Overlap: project Wilson eigenvalues onto circle
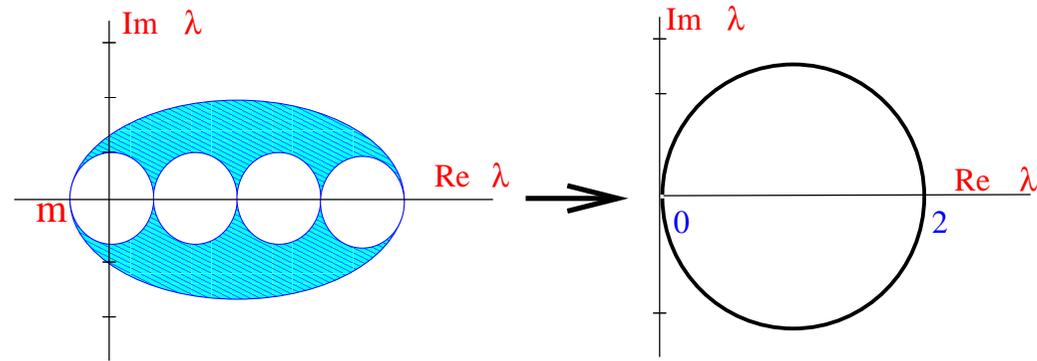
- $D = 1 + V$
- $V = (D_W D_W^\dagger)^{-1/2} D_W$
- $V^\dagger V = 1$ Ginsparg-Wilson condition
- normality restored
- $m < 0$: Wilson hopping parameter "supercritical"

Exact chiral symmetry

$$\psi \to e^{i\theta\gamma_5}\psi$$

$$\overline{\psi} \to \overline{\psi}e^{i\theta\hat{\gamma}_5}$$

$$\hat{\gamma}_5 = -V\gamma_5$$

$$\nu = \frac{1}{2}\mathrm{Tr}\hat{\gamma}_5$$
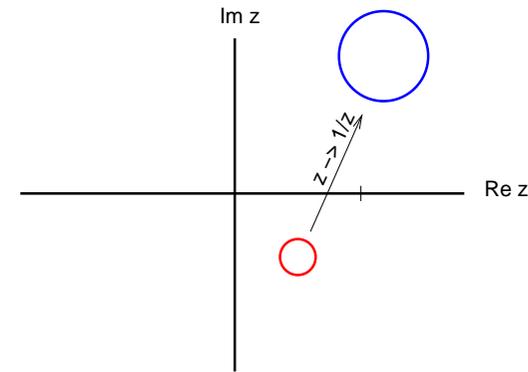
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A Cheshire Chiral Condensate

Consider the overlap

- Eigenvalues in complex conjugate pairs on a circle

  - $D = 1 + V$

  - $V^\dagger V = 1$
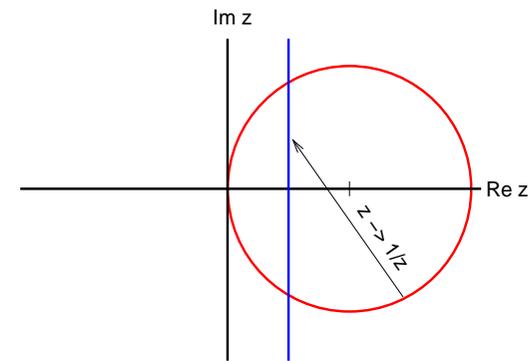
- Calculate the condensate

$$\langle\overline{\psi}\psi\rangle = \langle\mathrm{Tr}D^{-1}\rangle = \left\langle\sum\frac{1}{\lambda_i}\right\rangle = \left\langle\sum\mathrm{Re}\frac{1}{\lambda_i}\right\rangle$$

Inverting a complex circle gives another circle

Circle for D touches the origin
- inverses collapse onto line $\mathrm{Re}\frac{1}{\lambda} = \frac{1}{2}$
- For all eigenvalues!

For the condensate

$$\langle\overline{\psi}\psi\rangle = \sum \mathrm{Re}\frac{1}{\lambda_i} = \sum \frac{1}{2} = \frac{N}{2} \neq 0$$

- N is the dimension of D

- Independent of any dynamics!?

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Do we have the wrong operator?

- $\overline{\psi}\psi$ nontrivial under generalized chiral symmetry
- is $\langle\overline{\psi}(1 - D/2)\psi\rangle$ better?
  - goes to its negative on chiral rotation

The second term is also easy to calculate

$$\langle\overline{\psi}D\psi\rangle = \mathrm{Tr}D^{-1}D = \mathrm{Tr}1 = N$$

Combining:

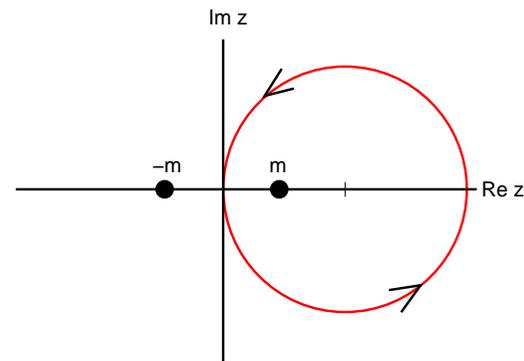$$\langle\overline{\psi}(1 - D/2)\psi\rangle = N/2 - N/2 = 0$$

Oops, the condensate is gone?

Resolution: $V \to \infty$ and $m \to 0$ limits don't commute

- add a small mass
- $\langle \overline{\psi} \psi \rangle = \sum \frac{1}{\lambda + m}$
- look for a jump as m passes through zero

Contour integral around the GW circle $\lambda = 1 + e^{i\theta}$

$$i \int_0^{2\pi} d\theta \frac{\rho(\theta)}{1 + e^{i\theta} + m}$$

- pole at $-m$ moves from inside to outside the circle
- residue $\rho(0) \equiv \lim_{\theta \to 0} \rho(\theta)$
- integral jumps by $2\pi\rho(0)$

The Banks-Casher relation for the overlap

Another Puzzle

Two flavors

- expect spontaneous chiral symmetry breaking, light pions
- should have $\rho(0) \neq 0$

One flavor

- anomaly breaks all chiral symmetry
- $\langle \overline{\psi}\psi \rangle$ behaves smoothly at $m \sim 0$
- should have $\rho(0) = 0$
  - note: zero modes give smooth contribution to $\langle \overline{\psi}\psi \rangle$ (see later)

But

- one flavor has one power of $|D|$ $\qquad\qquad$ $Z = \int (dA)\, |D|^1\, e^{-S_G}$
- two flavors have two powers $\qquad\qquad\qquad$ $Z = \int (dA)\, |D|^2\, e^{-S_G}$
- Two flavors should naively supress small eigenvalues more!

How can two flavors have the bigger $\rho(0)$???

ρ depends on distribution of A depends on ρ

Not just low eigenvalues are relevant

- fermions tend to smooth out gauge fields

- more fermions smooth things more

- involves all scales

- smoother fields give more low eigenvalues

- overcomes suppression from more powers of the determinant

$$\int dA \, |D|^{N_f} \, e^{-S_g(A)}$$

Increasing N_f can increase density of small eigenvalues!

Zero modes?

Again insert a small mass

$$Z = \int dA\, e^{-S_g} \prod (\lambda_i + m)$$

As m goes to zero any configurations involving a $\lambda = 0$ drop out

- are "instantons" irrelevant in the chiral limit?

No: add sources $\eta, \overline{\eta}$

$$Z(\eta, \overline{\eta}) = \int dA \, d\psi \, d\overline{\psi} \, e^{-S_g + \overline{\psi}(D+m)\psi + \overline{\psi}\eta + \overline{\eta}\psi}$$

- integrate out fermions

$$Z = \int dA \, e^{-S_g + \overline{\eta}(D+m)^{-1}\eta} \prod (\lambda_i + m)$$

If source overlaps with the zero mode eigenvector $(\psi_0, \eta) \neq 0$
- $1/m$ in source term cancels m from determinant
- With multiple flavors
  - need source factor from each flavor: "t'Hooft vertex"

Instantons drop out of Z
- but survive in correlation functions
  - small mass extrapolations are numerically difficult

Masses and topology

One massless flavor

- 't Hooft vertex quadratic in fermion fields
- generates smooth contribution to $\langle \overline{\psi}\psi \rangle$
  - $1/m$ term in $\sum 1/\lambda_i$ cancels m from $|D|$
- an additive mass shift   "renormalon"
  - non-perturbative
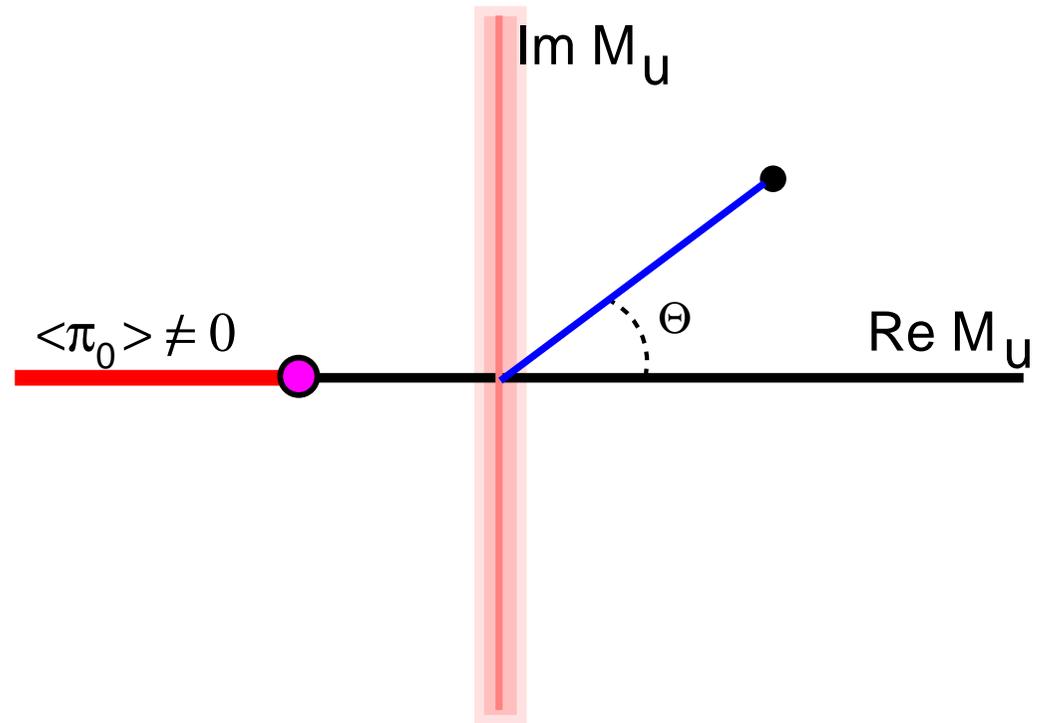 - depends on scale and regulator

Overlap operator is not unique

- depends on
  - particular input D chosen
  - Wilson mass (domain wall height)

Scheme dependent additive mass shift

- $m = 0$ is not a physical concept for a single flavor
- $m_u = 0$ cannot solve strong CP problem

Conventional variables

Im M_u / Re M_u figure with $\langle \pi_0 \rangle \neq 0$ and angle Θ

- Λ

- $|M|$

- $\theta :$ $\tan(\theta) = \frac{\mathrm{Im}\,M}{\mathrm{Re}\,M}$

With one flavor these are singular coordinates
  - scheme dependent additive shift in $\mathrm{Re}\,M$ changes θ

$\mathrm{Re}\,M$ and $\mathrm{Im}\,M$ are better independent variables
  - CP symmetry protects the real axis
  - imaginary axis can shift
    - Di Vecchia and Veneziano (1980)

$m = 0 \longleftrightarrow$ **vanishing topological susceptibility?**

Winding number ambiguous when $D_W D_W^\dagger$ not invertible

- occurs with eigenvalues near domain wall height

Admissibility condition

- strong constraint on allowed plaquettes

- disallows rough configurations, making winding unique

- Violates reflection positivity!

Is the topological susceptibility a well defined observable?

- do we care?

Final remarks

Eigenvalues can give some insight

- Banks-Casher

But can be misleading

- adding flavors enhances low eigenvalues

Unresolved issues

- do we understand non-perturbative ambiguities?
    - is topological susceptibility an observable?
    - are rough gauge fields essential?
- how do these issues interplay with quark masses?
    - is $m_u = 0$ a definable concept?