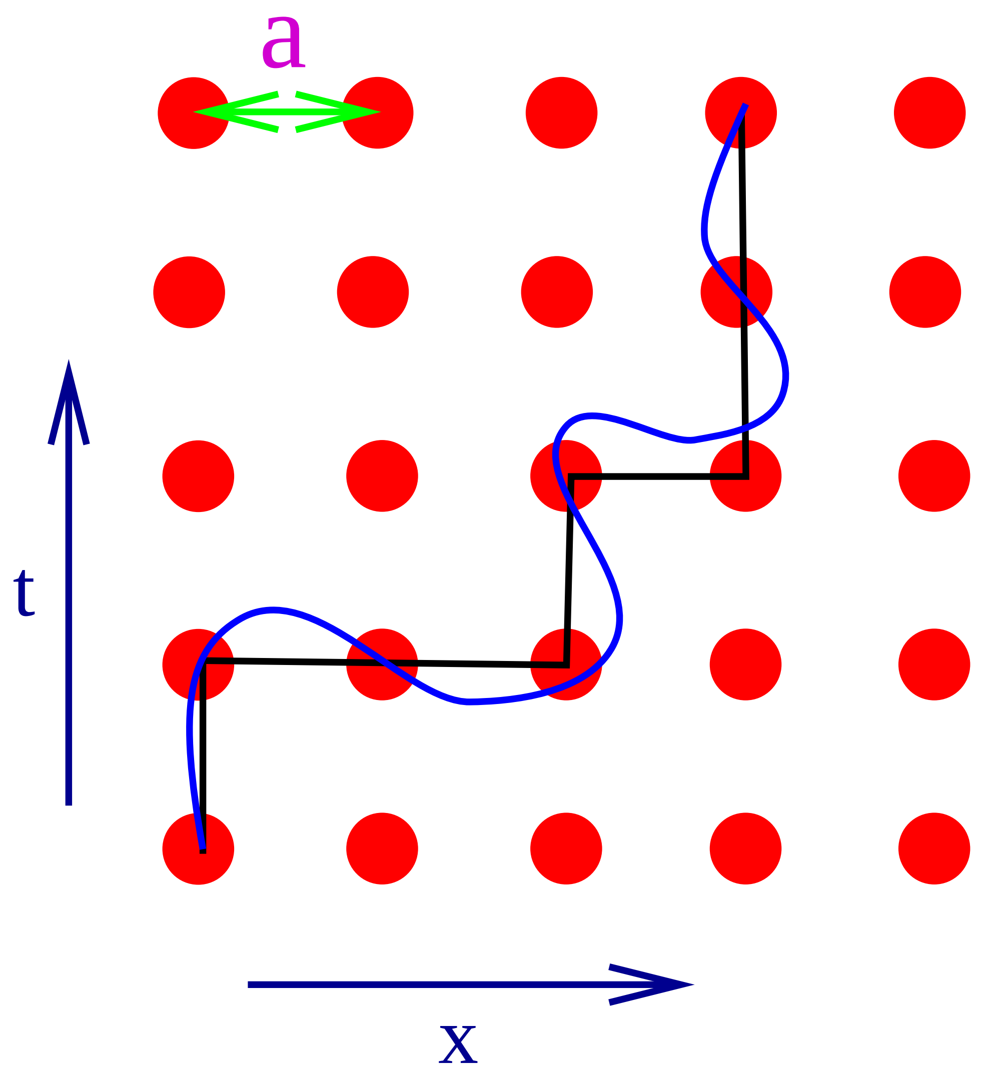
Lattice Fermions



Lattice Fermions

Consider some lattice Dirac operator D

- assume gamma five hermiticity $\gamma_5 D \gamma_5 = D^\dagger$
- all operators in practice satisfy this (except twisted mass)

Divide D into hermitean and antihermitean parts

$$D = K + M$$

$$K = (D - D^\dagger)/2$$

$$M = (D + D^\dagger)/2$$

Then by construction

$$[K, \gamma_5]_+ = 0$$

$$[M, \gamma_5]_- = 0$$

On a lattice everything is finite; so $\mathrm{Tr}\gamma_5 = 0$

$M \to e^{i\theta\gamma_5} M$ is an exact symmetry of the determinant

$$|K + M| = |e^{i\gamma_5\theta/2}(K + M)e^{i\gamma_5\theta/2}| = |K + e^{i\theta\gamma_5} M|$$

Where did the anomaly hide?

This must be a flavored chiral symmetry

All lattice actions bring in extra structure

Naive and staggered fermions have doublers

- half use γ_5 and half $-\gamma_5$
- the naive chiral symmetry is actually flavored

Wilson and overlap fermions

- M is not a constant
- heavy states appear to cancel the anomaly
- chiral symmetry modified by their mass

Continuum free fermion action density

$$\overline{\psi}D\psi = \overline{\psi}(\not{\partial} + m)\psi$$

in momentum space

$$\overline{\psi}(i\not{p} + m)\psi.$$

D has both Hermitean and anti-Hermitean parts

- Hermitian mass term is a constant
- this won't be the case on the lattice

Lattice transcription replaces p with trigonometric functions

- Fourier transform

$$\tilde{\phi}_q = \sum_j e^{-iqj}\phi_j \qquad\qquad \phi_j = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iqj}\tilde{\phi}_q$$

- $\sum \phi^*_{j+1}\phi_j - \phi^*_j\phi_{j+1} = -2i \int_{-\pi}^{\pi} \frac{dq}{2\pi} \sin(q)\tilde{\phi}^*(q)\tilde{\phi}(q)$

Naive lattice fermions

- replace ∂_μ with nearest neighbor differences

$$\overline{\psi}\left(\frac{i}{a}\sum_\mu \gamma_\mu \sin(p_\mu a) + m\right)\psi$$

- at small p goes to desired $\overline{\psi}(i\gamma_\mu p_\mu + m)\psi$

But this also has low energy excitations for $p_\mu \sim \pi/a$

- we have $2^4 = 16$ “doublers”

Wilson: Add a momentum dependent mass

$$\overline{\psi} D_W \psi = \overline{\psi} \left(\frac{1}{a} \sum_\mu (i\gamma_\mu \sin(p_\mu a) + 1 - \cos(p_\mu a)) + m \right) \psi.$$

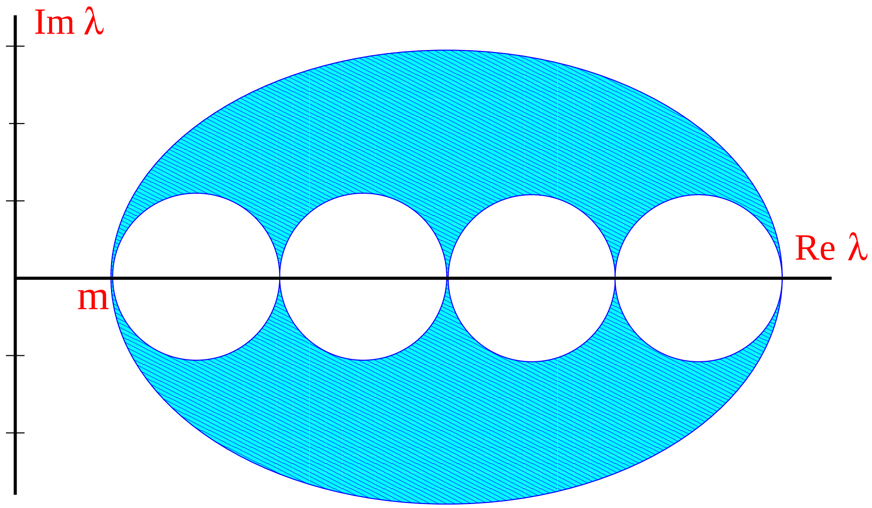
- for small momentum $\frac{1}{a}(1 - \cos(p_\mu a)) = O(a)$
- for momentum components near π the eigenvalues are of order $1/a$

Eigenvalues for the free theory at

$$\lambda = \pm \frac{i}{a} \sqrt{\sum_\mu \sin^2(p_\mu a)} + \frac{1}{a} \sum_\mu (1 - \cos(p_\mu a)) + m.$$

- both real and imaginary parts even at $m = 0$

The eigenvalues form a set of “nested circles”



Notes

- $m \leftrightarrow -m$ not a symmetry
- essential for quantum theory with N_f odd
- chiral symmetry broken: $[D, \gamma_5]_+ \neq 0$
- m can get an additive renormalization

The Nielsen-Ninomiya theorem

Doubling closely tied to topology in momentum space

- Consider the gamma matrix convention

$$\vec{\gamma} = \sigma_1 \otimes \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$\gamma_0 = \sigma_2 \otimes I = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\gamma_5 = \sigma_3 \otimes I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Consider some anti-Hermitean Dirac operator D anti-commuting with γ_5

$$D = -D^\dagger = -\gamma_5 D \gamma_5.$$

Go to momentum space: D(p) a 4×4 matrix function of p_μ

The most general form is

$$D(p) = \begin{pmatrix} 0 & z(p) \\ -z^*(p) & 0 \end{pmatrix}$$

- where $z(p)$ is a quaternion

$$z(p) = z_0(p) + i\vec{\sigma} \cdot \vec{z}(p).$$

Any chirally symmetric Dirac operator

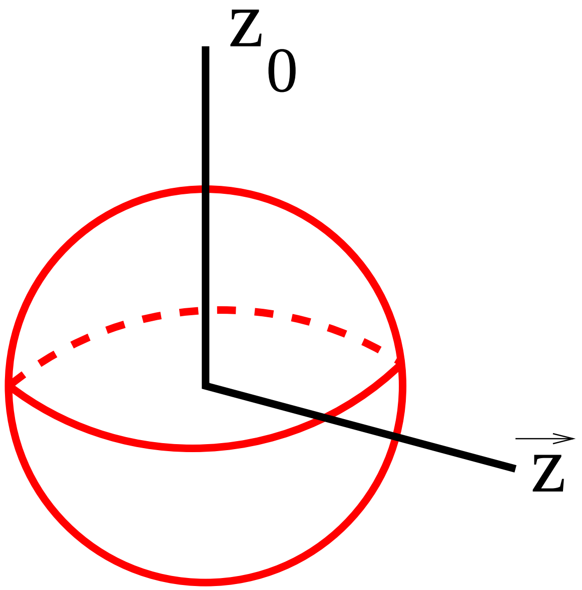
- maps momentum space onto the space of quaternions

Dirac equation expands $D(p)$ around a zero

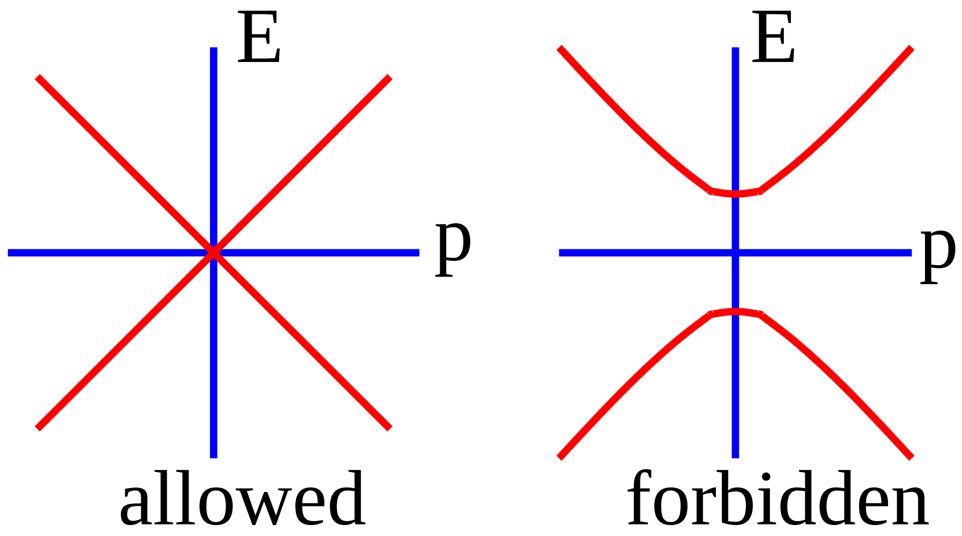
- $D(p) \simeq i\not{p} = i\gamma_\mu p_\mu$

Consider sphere of constant $|p|$ surrounding a zero

- this maps z non-trivially around the origin in quaternion space



Non-trivial mapping makes the zeros robust

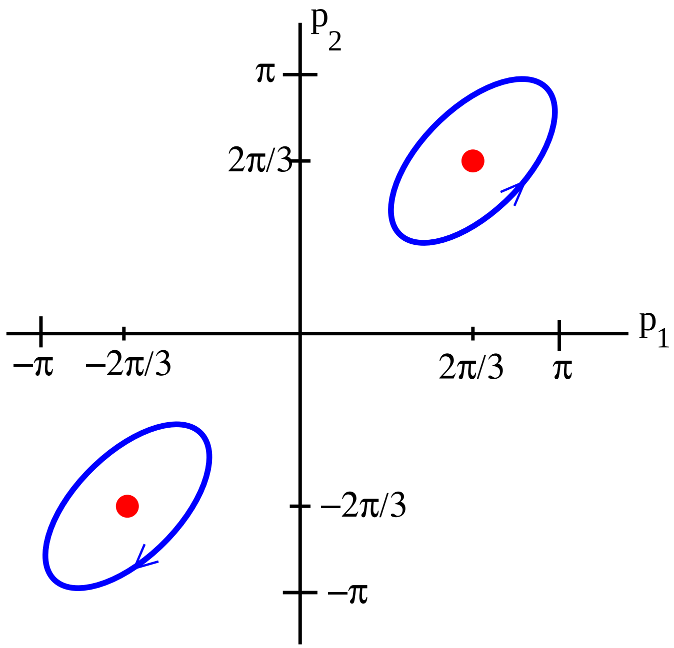


- masslessness is protected

Momentum space periodic over a Brilloin zone $-\pi \leq p_\mu < \pi$

- must have $z(p) = z(p + 2\pi n)$

Any wrapping must unwrap elswhere



Naive 16 doublers divide into two groups of 8

- with opposite mappings

An exact chiral lattice theory must have an even number of species

Minimal doubling

Several local chiral lattice actions with $N_f = 2$ flavors are known

- symmetry preserves multiplicative mass renormalization

Known variations break hyper-cubic symmetry

- introduces anisotripic counter terms

A single local field can create two species

Karsten Wilczek example

- replace Wilson term with imaginary chemical potential
- the free momentum space Dirac operator

$$D(p) = i\sum_{i=1}^{3}\gamma_i \sin(p_i) + \frac{i\gamma_4}{\sin(\alpha)}\left(\cos(\alpha) + 3 - \sum_{\mu=1}^{4}\cos(p_\mu)\right)$$

- exact chiral symmetry: $[D,\gamma_5]_+ = 0$

Propagator has two poles, at $\vec{p} = 0$, $p_4 = \pm\alpha$

- parameter α allows adjusting the relative pole positions
- original form used $\alpha = \pi/2$, $\cos(\alpha) = 0$

Karsten Wilczek action maintains one exact chiral symmetry

$$[D, \gamma_5]_+ = 0$$

The two species are not equivalent

- they have opposite chirality
- expand the propagator around the poles

  around $p_4 = +\alpha$ uses the usual gamma matrices

  $p_4 = -\alpha$ uses $\gamma'_\mu = \Gamma^{-1}\gamma_\mu\Gamma$

  for this action $\Gamma = i\gamma_4\gamma_5$

  $\gamma'_5 = -\gamma_5$
- exact chiral symmetry is "flavored"

  like continuum $\tau_3\gamma_5$

Inserting gauge fields

$$D_{ij} = U_{ij} \sum_{\mu=1}^{3} \gamma_i \frac{\delta_{i,j+e_\mu} - \delta_{i,j-e_\mu}}{2}$$
$$+ \frac{i\gamma_4}{\sin(\alpha)} \left((\cos(\alpha) + 3)\delta_{ij} - U_{ij} \sum_{\mu=1}^{4} \frac{\delta_{i,j+e_\mu} + \delta_{i,j-e_\mu}}{2} \right).$$

Broken hypercubic symmetry can induce asymmetry

- renormalization of α
- renormalization of temporal hopping
- renormalization of time-like plaquettes

Issues with controlling these counter-terms await simulations

A single field ψ can create either of the two species

Can separate them with point splitting

Consider for the free theory

$$u(q) = \frac{1}{2}\left(1 + \frac{\sin(q_4 + \alpha)}{\sin(\alpha)}\right)\psi(q + \alpha e_4)$$
$$d(q) = \frac{1}{2}\,\Gamma\left(1 - \frac{\sin(q_4 - \alpha)}{\sin(\alpha)}\right)\psi(q - \alpha e_4)$$

- here $\Gamma = i\gamma_4\gamma_5$ for the Karsten/Wilczek action

  accounts for two species useing different gammas
- construction not unique

Inserting gauge field factors in position space

$$u_x = \frac{1}{2} e^{i\alpha x_4} \left(\psi_x + i \; \frac{U_{x,x-e_4}\psi_{x-e_4} - U_{x,x+e_4}\psi_{x+e_4}}{2\sin(\alpha)} \right)$$
$$d_x = \frac{1}{2} \Gamma e^{-i\alpha x_4} \left(\psi_x - i \; \frac{U_{x,x-e_4}\psi_{x-e_4} - U_{x,x+e_4}\psi_{x+e_4}}{2\sin(\alpha)} \right).$$

- phases remove oscillations from poles at non-zero momentum

Domain wall and overlap fermions

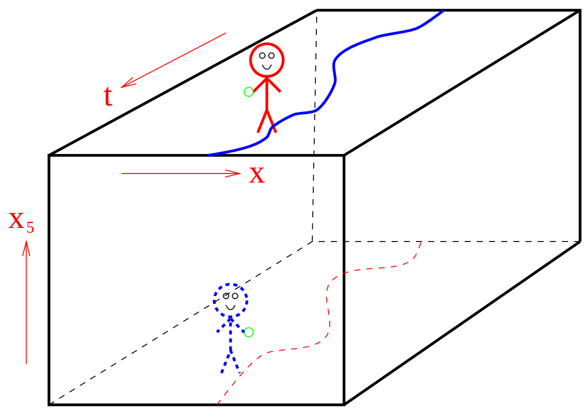
Wilson fermions with $K > K_c$ have low energy surface modes

- naturally chiral
- mixing through tunnelling between walls
- example of a "topological insulator"

  robust conductiity only on surfaces

Use this for "domain wall" fermions

- work on a 5-d lattice of finite size $0 \leq x_5 < l_s$
- identify surface modes with physical quarks
- bulk modes go to infinite mass in continuum limit



Overlap fermion limit

- drive bulk modes to large energy
- take $l_s \to \infty$
- name from ground state "overlap" of 5d transfer matrices

Can be reformulated directly in four dimensions

- possess an order a modified chiral symmetry

$$\psi \longrightarrow e^{i\theta\gamma_5}\psi$$
$$\overline{\psi} \longrightarrow \overline{\psi}e^{i\theta(1-aD)\gamma_5}$$

- also maintain $\gamma_5 D \gamma_5 = D^\dagger$

Note the asymmetric treament of ψ and $\overline{\psi}$

Invariance of $\overline{\psi} D \psi$ implies

$$D\gamma_5 = -\gamma_5 D + aD\gamma_5 D = -\hat{\gamma}_5 D$$

- where $\hat{\gamma}_5 \equiv (1 - aD)\gamma_5$.
- this is the "Ginsparg-Wilson relation"
- naive anticommutation corrected by $O(a)$

The GW relation is equivalent to the unitarity of

$$V = 1 - aD$$

GW $\longrightarrow$ $V^\dagger V = 1$

Neuberger: construct V via unitarization

- start with an undoubled Dirac operator
- i.e. the Wilson operator D_w

$$V = -D_w(D_w^\dagger D_w)^{-1/2}.$$

- to define $(D_w^\dagger D_w)^{-1/2}$

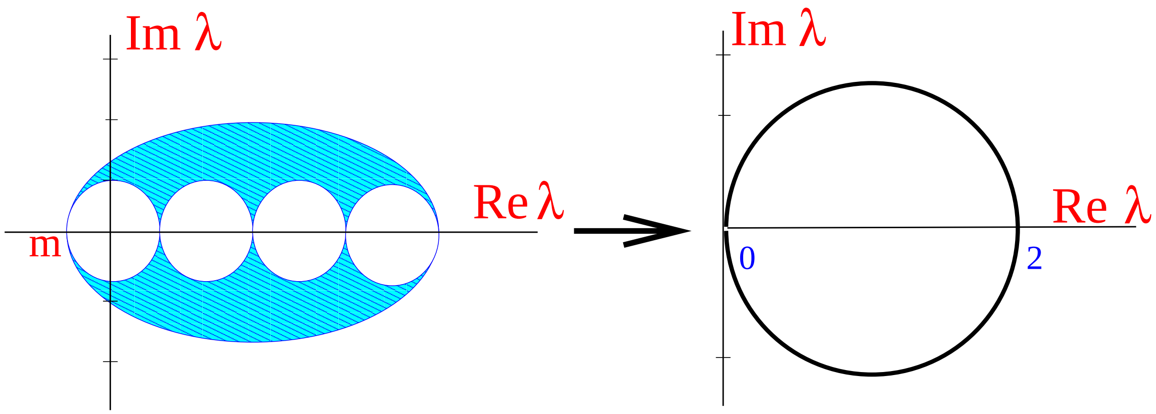
  1) diagonalize $D_w^\dagger D_w$

  2) take the square root of the eigenvalues

  3) undo the diagonalizing unitary transformation.

Given V, the overlap operator is

$$D = (1 - V)/a.$$



Properties of the overlap

- computationally demanding
- “normal:” $[D^\dagger, D] = 0$, unlike Wilson
- $\gamma_5 D \gamma_5 = D^\dagger$
- a modified exact chiral symmetry

$$\psi \to e^{i\theta\gamma_5}\psi$$

$$\overline{\psi} \to \overline{\psi} e^{i\theta\hat{\gamma}_5}$$

- where $\hat{\gamma}_5 = V\gamma_5$.

As with γ_5, $\hat{\gamma}_5$ is Hermitean and $\hat{\gamma}^2 = 1$

- all eigenvalues are ± 1
- defines an index

$$\nu = \frac{1}{2}\mathrm{Tr}\hat{\gamma}_5 = \mathrm{Tr}\frac{\gamma_5 + \hat{\gamma}_5}{2}$$

D has ν exact zero eigenvalues

- agrees with continuum index for smooth fields
- $1/2$ compensates real modes on opposite side of the unitarity circle

Some issues

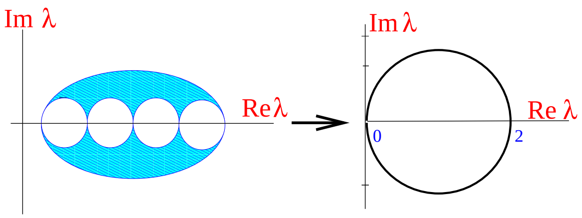
- The “kernel” projected for the overlap is somewhat arbitrary

  with D_w, dependence on the hopping parameter

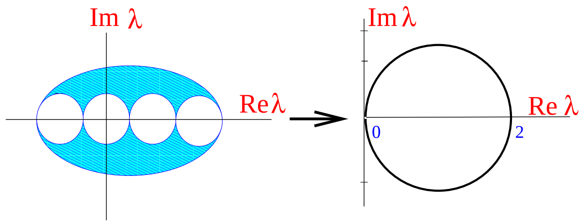
  need $K > K_c$ so one species projects out

  need K below doubler masses or too many massless fermions

- if $K < K_c$ still satisfy GW but no massless particles



$$N_f = 0$$



$$N_5 = 5$$

GW does NOT require massless Goldstone bosons

Issues continued

- ν can depend on choice of K

  depends on eigenvalue density in first circle

  more on this in the next lecture

- how local is the overlap?

  inversion destroys sparsity

  is the non-locality exponential?

Issues continued

- the one flavor theory

  no jump in condensate at vanishing mass

  gap in eigenvalue spectrum at zero?

- fermions not in the fundamental representation

  zero mode counting can fail on rough configurations

Staggered fermions

Spin “emerges” from a spinless field, like graphene in 2D

- less useful in practice since 4 doublers per flavor remain

Consider spinless fermions hopping in a background Z_2 gauge field

- color index but only a one component “spinor”

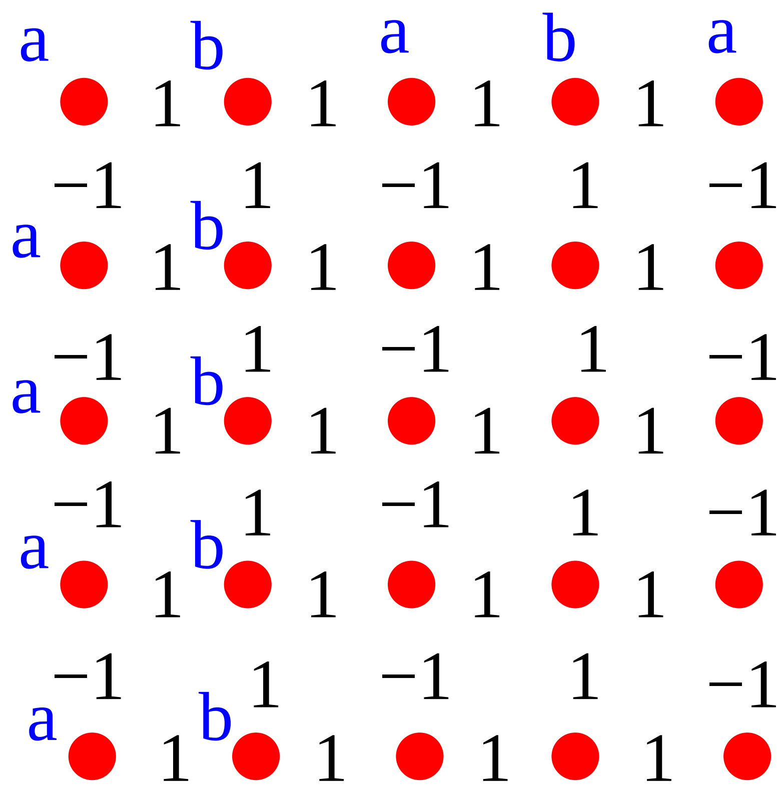
Drive every Z_2 plaquette to -1

- thread half integer magnetic flux through every plaquette

$$\beta_{Z_2} = -\infty$$

- one gauge choice puts phase factors on links as

$$Z_x = 1,\ Z_y = (-1)^x,\ Z_z = (-1)^{x+y},\ Z_t = (-1)^{x+y+z}$$



Translation invariance

- by 1 in t direction
- by 2 in x, y, z directions

  $2^3 = 8$ distinct types of site (2 in two dimensions) in momentum space, translate to a four component base spinor

$$\begin{pmatrix} \psi(0,0,0,0) \\ \psi(1,0,0,0) \\ \psi(0,1,0,0) \\ \psi(1,1,0,0) \\ \psi(0,0,1,0) \\ \psi(1,0,1,0) \\ \psi(0,1,1,0) \\ \psi(1,1,1,0) \end{pmatrix}$$

- eigenvalues of D proportional to

$$\pm i\sqrt{\cos^2(p_x) + \cos^2(p_y) + \cos^2(p_z) + \cos^2(p_t)}$$

Translation symmetry is by two in spatial directions

restricts $\vec{p}$ components to "half" zones $0 \le p_j < \pi$

temporal momentum has a full zone $-\pi \le p_t < \pi$

- 8 component spinor with two zeros at $(+\pi/2, +\pi/2, +\pi/2, \pm\pi/2)$

4 effective Dirac species species (tastes)

Chiral symmetry

- only nearest neighbor hopping

action anticommutes with $(-1)^{x+y+z+t} \sim$ "γ_5"

- Dirac "cones" come in each chirality
- a four "flavor" theory with one exact chiral symmetry

actually a "flavored" chiral symmetry

consistent with anomaly

These are staggered fermions

- spin emerges dynamically for a one component field

  Z_2 background field at "$\beta = -\infty$" gives sign factors

- inherently multiple degenerate species

- rooting issues discussed in previous and next lecture

Wilson fermions, the Aoki phase, and twisted mass

At finite lattice spacing with two degenerate Wilson fermions

- lattice artifacts can generate spontaneous flavor and CP violation
- over a finite range of hopping parameters $K \sim K_{cr}$

  two Goldstone bosons from flavor breaking

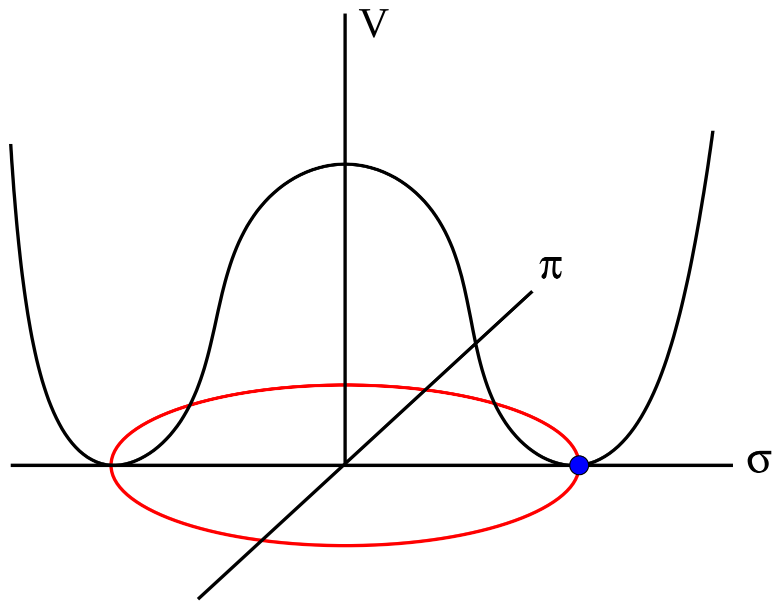
  a third massless state at a second order boundary

Aoki (1983): these become the three pions in the continuum limit

A simple picture built on the linear sigma model:

MC (1996), Sharpe & Singleton (1998)

$$V(\sigma, \vec{\pi}) = \lambda \left(\sigma^2 + \vec{\pi}^2 - v^2\right)^2$$



- pions massless because of flat directions

Wilson fermions introduce lattice artifacts

- chiral symmetry broken, model corrections

$$V(\sigma, \vec{\pi}) = \lambda(\sigma^2 + \vec{\pi}^2 - v^2)^2 - m\sigma + c_1\sigma + c_2\sigma^2 + \ldots$$

- c_1 correction additively renormalizes mass

  tune hopping to remove

  an additive mass renormalization

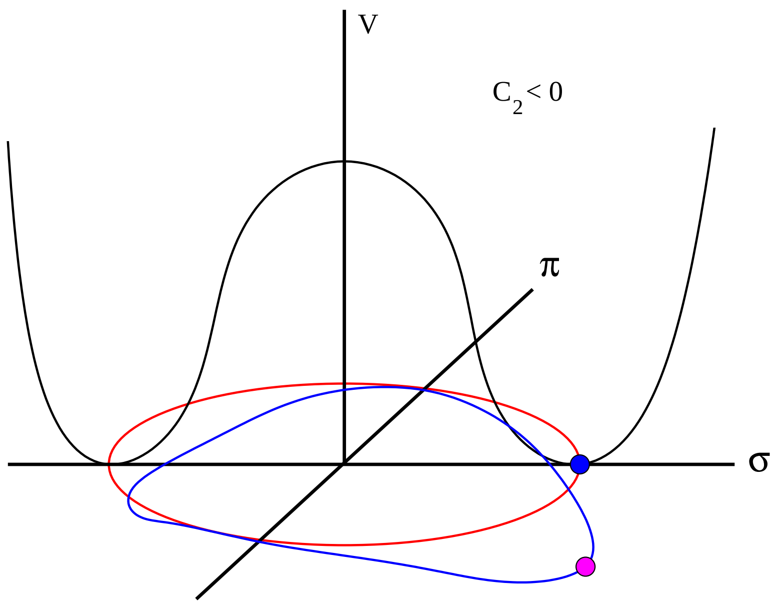
  critical hopping $K_{cr}(\beta)$ moves away from $1/8$

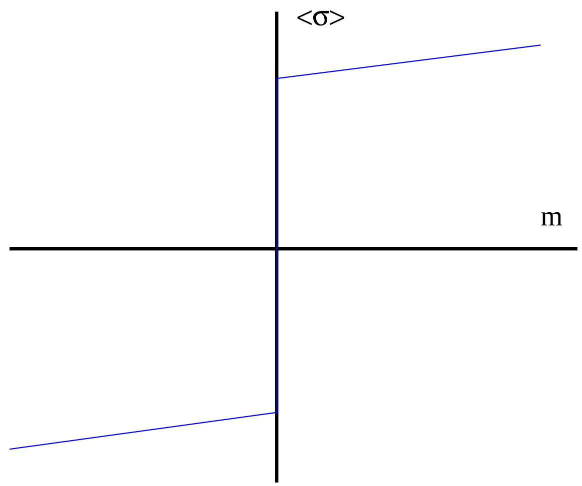
$c_2\sigma^2$ distorts potential quadratically

- sign of c_2 can depend on gauge action

$c_2 < 0$ chiral transition goes first order

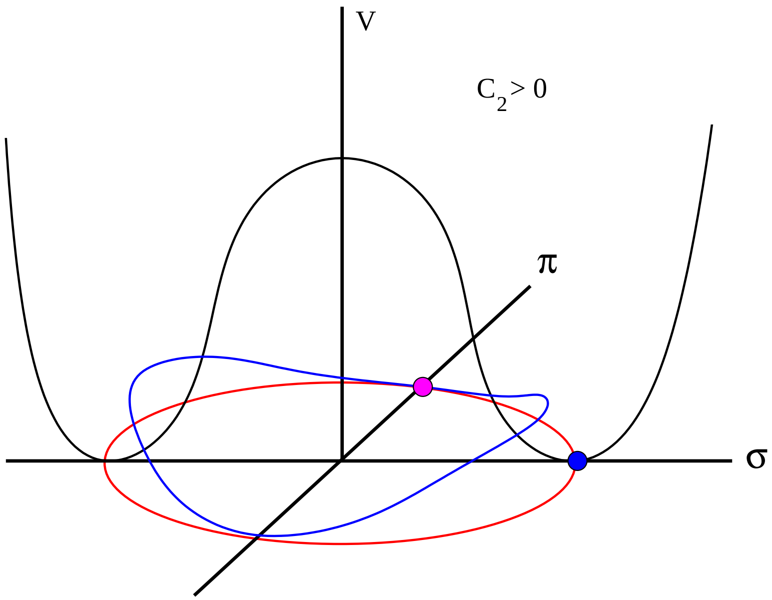
- no exact Goldstone bosons $\qquad m_\pi^2 \sim |c_2|$

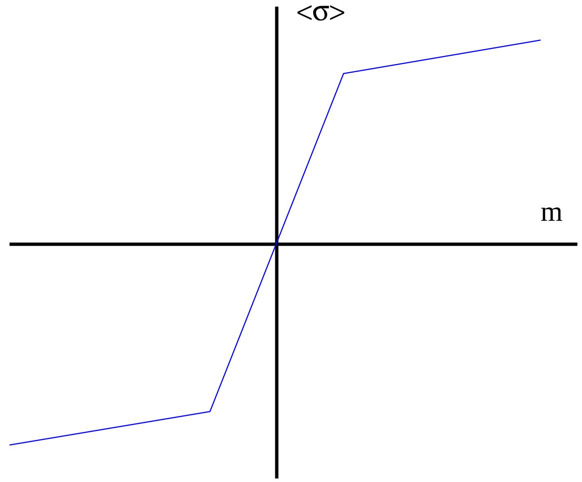




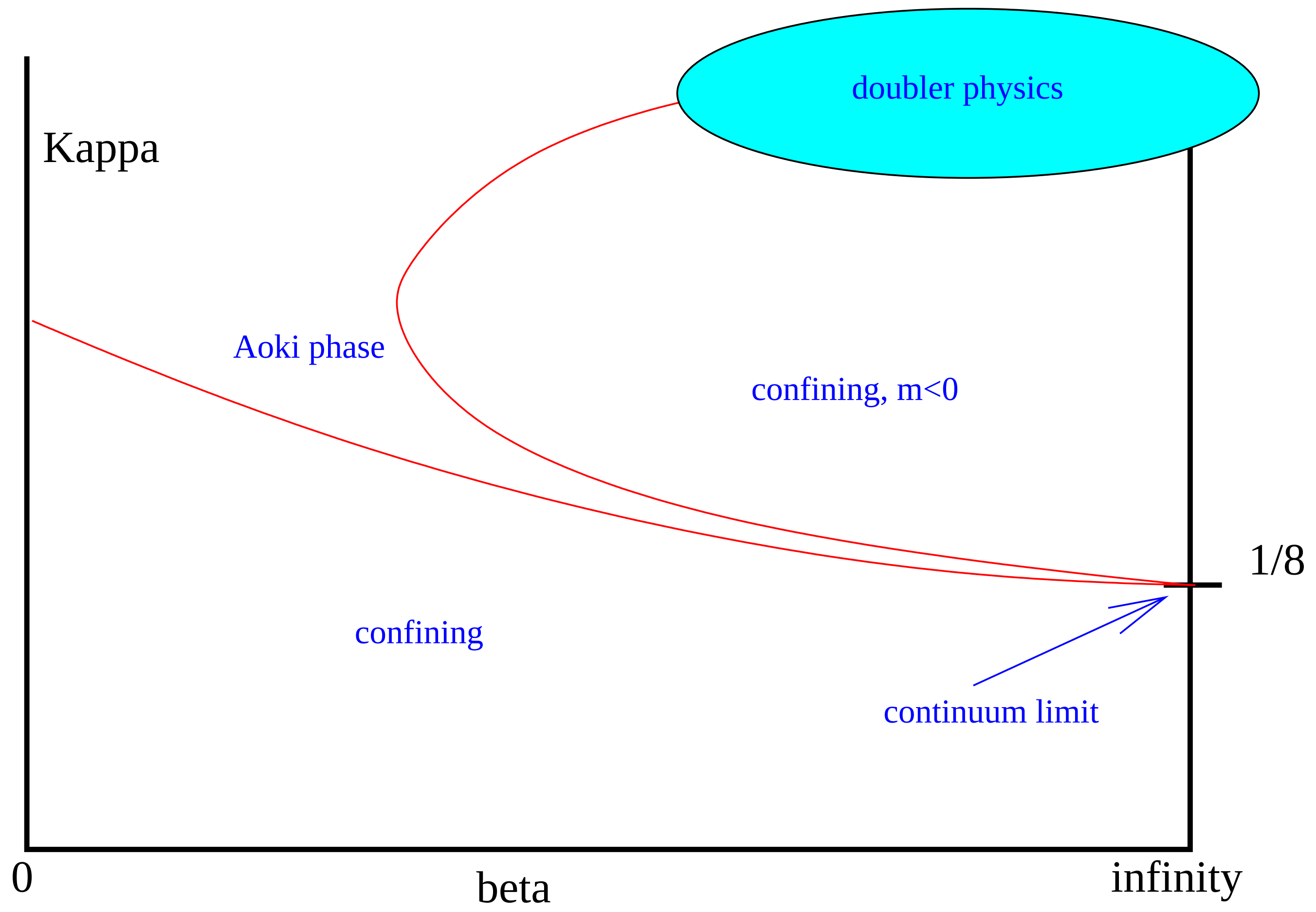
If $c_2 > 0$ chiral transition splits into two second order transitions

- separated by phase with $\langle\vec{\pi}\rangle \neq 0$
  - breaks parity and flavor spontaneously
  - two Goldstone bosons from flavor breaking
  - third massless pion only at critical point
- the "Aoki phase"





The canonical picture with $c_2 > 0$



$(\kappa \sim \frac{1}{m+8})$

The c_2 term breaks the equivalence of different chiral directions

- no longer equivalent physics with

  $m\sigma \to m\cos(\theta)\sigma + m\sin(\theta)\pi_3$

- rotation gives up and down quarks opposite phases

  $m_u \to e^{i\theta} m_u \qquad m_d \to e^{-i\theta} m_d$

  phases cancel in CP parameter Theta

Suggests on the lattice a new "twisted mass" term

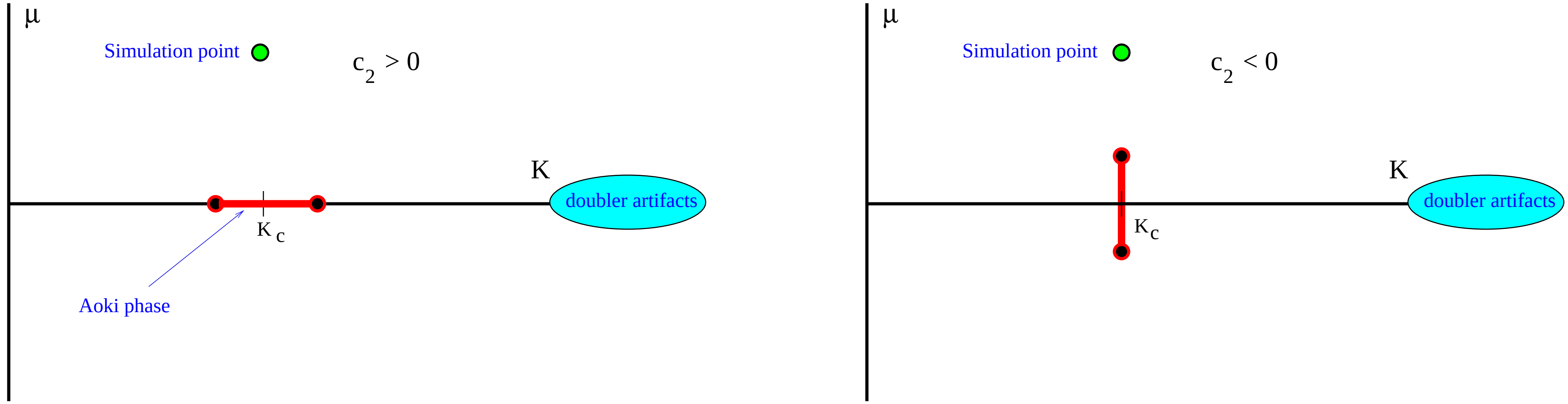
$$V(\sigma, \vec{\pi}) = \lambda(\sigma^2 + \vec{\pi}^2 - v^2)^2 - m\sigma + c_2\sigma^2 - \mu\pi_3$$

c_1 absorbed in m

μ: "Magnetic field" conjugate to the Aoki phase order parameter

Motivations for twisted mass:

- $O(a)$ lattice artifacts can be tuned to cancel
- fermion determinant remains positive
- faster than overlap or domain wall
- allows continuation around Aoki phase



Which action is best?

Staggered: very fast, but too many species for QCD

Wilson: fast, but bad chiral properties

Twisted mass: fast, but still needs some tuning

Domain wall: some cost over Wilson, but improved chiral symmetry

Minimal doubling: counterterms need more study

Overlap: slow but elegant chiral properties

- All allow various "improvements" (smearing, . . .)
- All are in current use; pick your favorite