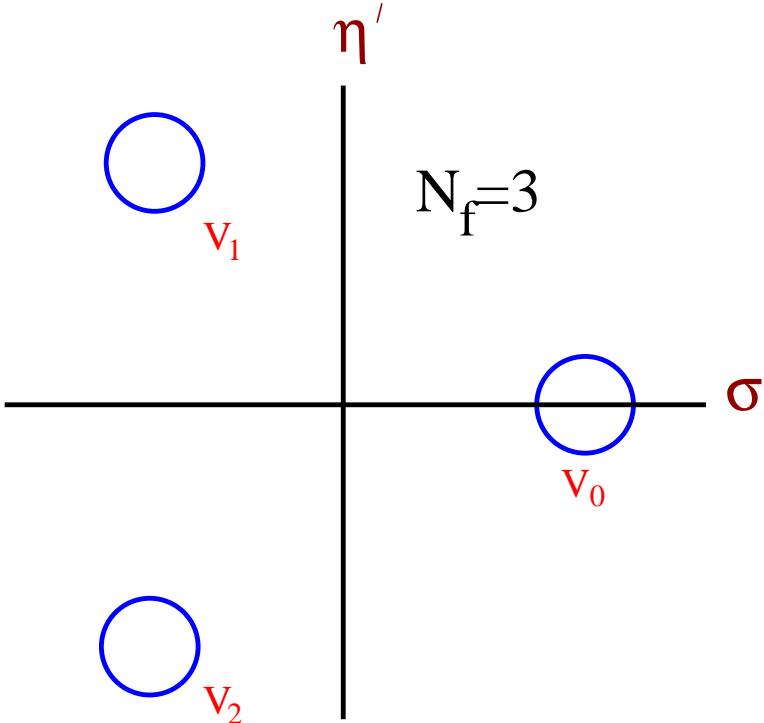
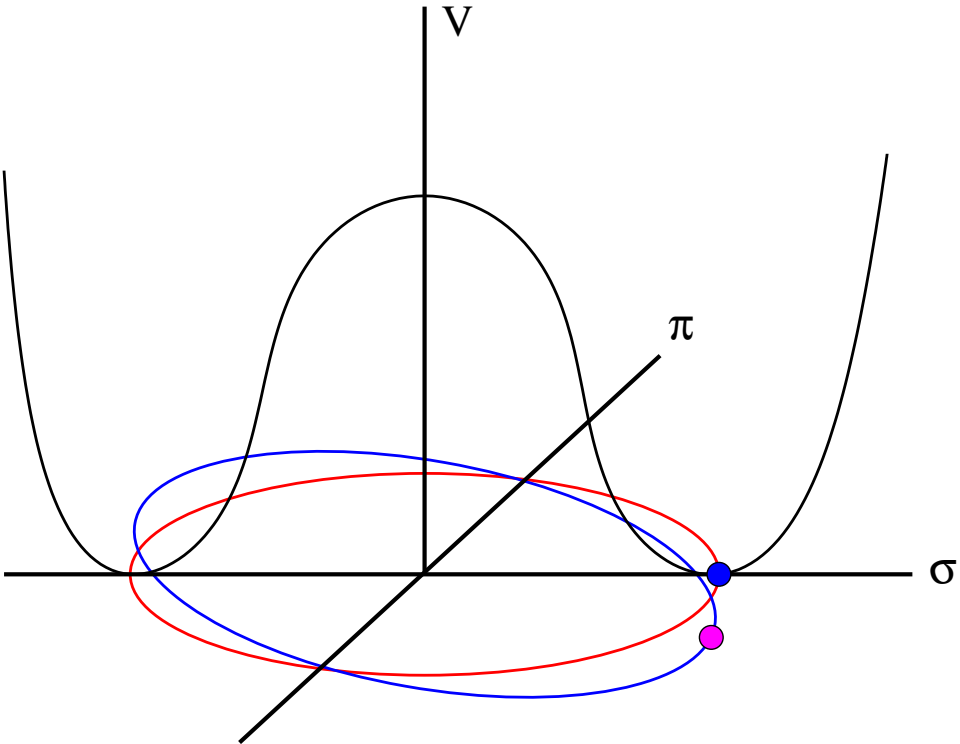


Lecture II: Chiral symmetries and anomalies

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BNL



Two powerful tools for non-perturbative strong interactions

- the lattice
 - rigorous but computationally demanding
- chiral symmetry
 - expansion in quark masses and momenta
 - qualitative understanding of low energy meson physics
 - provides checks on lattice ideas

Three sources of chiral symmetry breaking in QCD

- spontaneous breaking $\langle \bar{\psi}\psi \rangle \neq 0$
 - explains lightness of pions
- implicit breaking of $U(1)$ by the anomaly
 - explains why η' is not so light
- explicit breaking from quark masses
 - pions are not exactly massless

Rich physics from the interplay of these three effects



<http://cp3-origins.dk/a/9471>

Venerable issues

- Dashen, 1971: possible spontaneous strong CP violation
 - before QCD!
- 't Hooft, 1976: ties between anomaly and gauge field topology
- Fujikawa, 1979: fermion measure and the anomaly
- Witten, 1980: connections with effective Lagrangians
- MC 1995: Why is chiral symmetry so hard on the lattice

Consequences have raised some bitter controversies

Axial anomaly in N_f flavor massless QCD

- leaves behind a residual Z_{N_f} flavor-singlet chiral symmetry
- tied to gauge field topology and the QCD theta parameter

Consequences

- degenerate $m \neq 0$ quarks: first-order transition at $\Theta = \pi$
 - sign of mass relevant for odd N_f : perturbation theory incomplete
 - $N_f = 1$: no symmetry for mass protection
 - $m_u = 0$ cannot solve the strong CP problem
 - nontrivial N_f dependence:
 - invalidates rooting
- } controversial

Consider QCD with N_f light quarks and assume

- the field theory exists and confines
- spontaneous chiral symmetry breaking $\langle \bar{\psi}\psi \rangle \neq 0$
- $SU(N_f) \times SU(N_f)$ chiral perturbation theory makes sense
- anomaly gives η' a mass
- N_f small enough to avoid any conformal phase

Use continuum language

- imagine some non-perturbative regulator in place (lattice?)
 - momentum space cutoff much larger than Λ_{QCD}
 - lattice spacing a much smaller than $1/\Lambda_{QCD}$

Construct effective potential V for meson fields

- V represents vacuum energy density for a given field expectation
- formally via a Legendre transformation
- assume regulator allows defining composite fields

For simplicity initially consider

- degenerate quarks with small mass m
- N_f even
 - interesting subtleties for odd N_f

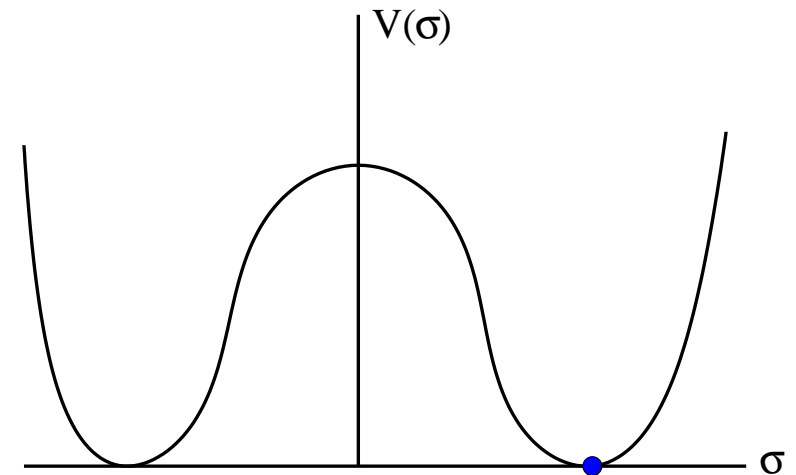
Work with composite fields

- $\sigma \sim \bar{\psi}\psi$
- $\pi_\alpha \sim i\bar{\psi}\lambda_\alpha\gamma_5\psi$
- $\eta' \sim i\bar{\psi}\gamma_5\psi$

λ_α : Gell-Mann matrices for $SU(N_f)$

Spontaneous symmetry breaking at $m = 0$

- $V(\sigma)$ has a double well structure
 - vacuum has $\langle\sigma\rangle = v \neq 0$
 - minimum of $V(\sigma) = \pm v$

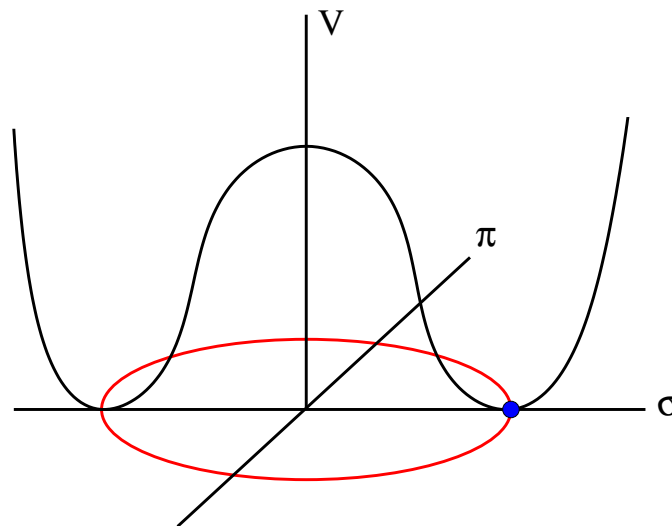


Ignore convexity issues

- phase separation occurs in a concave regions

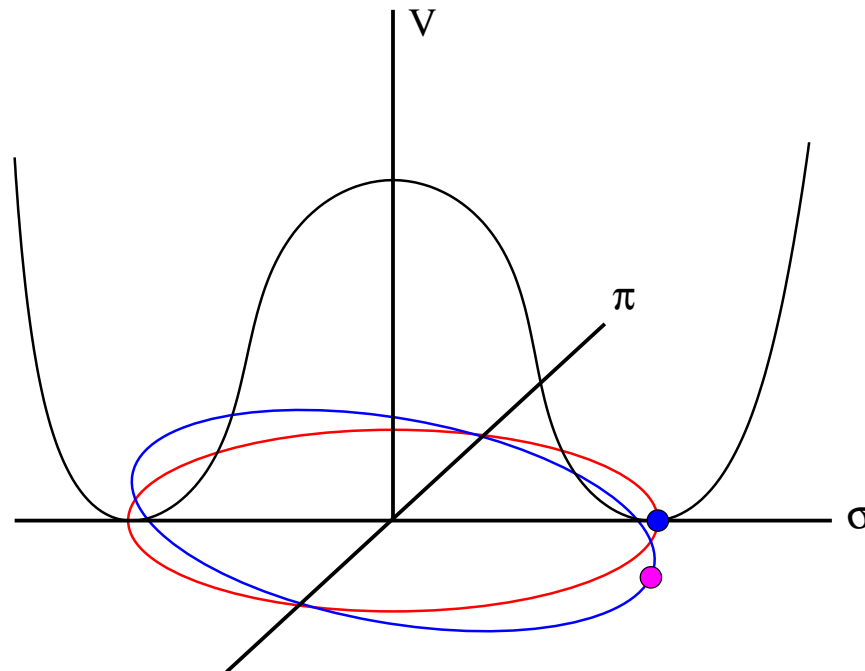
Nonsinglet pseudoscalars are Goldstone bosons

- massless theory symmetric under flavored chiral rotations
 - $\psi \rightarrow e^{i\phi\gamma_5\lambda^\alpha} \psi$
 - $\sigma \rightarrow \cos(\phi)\sigma + \sin(\phi)\pi^\alpha$
 - $\pi^\alpha \rightarrow \cos(\phi)\pi^\alpha - \sin(\phi)\sigma$ $(N_f = 2)$
- potential has $N_f^2 - 1$ “flat” directions
 - one for each generator of $SU(N_f)$



Small mass selects vacuum

- $V \rightarrow V - m\sigma$
- $\langle \sigma \rangle \sim +v \quad \langle \pi \rangle = 0$
- Goldstones acquire mass $\sim \sqrt{m}$



The vacuum is not empty

Particles are waves on the background

Anomaly gives the $\eta' \sim i\bar{\psi}\gamma_5\psi$ a mass even if $m_q = 0$

- $m_{\eta'} = O(\Lambda_{QCD})$
- $V(\sigma, \eta')$ **not** symmetric under
 - $\psi \rightarrow e^{i\phi\gamma_5}\psi$
 - $\sigma \rightarrow \sigma \cos(\phi) + \eta' \sin(\phi)$
 - $\eta' \rightarrow -\sigma \sin(\phi) + \eta' \cos(\phi)$

Expand the effective potential near the vacuum state $\sigma \sim v$ and $\eta' \sim 0$

- $V(\sigma, \eta') = m_\sigma^2(\sigma - v)^2 + m_{\eta'}^2\eta'^2 + O((\sigma - v)^3, \eta'^4)$
 - both masses of order Λ_{QCD}

In quark language

Classical symmetry $[\gamma_5, \mathcal{D}]_+ = 0$ implies

- $\psi \rightarrow e^{i\phi\gamma_5/2}\psi$
- $\bar{\psi} \rightarrow \bar{\psi}e^{i\phi\gamma_5/2}$
- mixes σ and η'
 - $\sigma \rightarrow \sigma \cos(\phi) + \eta' \sin(\phi)$
 - $\eta' \rightarrow -\sigma \sin(\phi) + \eta' \cos(\phi)$

This symmetry is “anomalous”

- any valid regulator must break chiral symmetry
 - remnant of the breaking survives in the continuum

Fujikawa: Variable change alters fermion measure

- $d\psi \rightarrow |e^{-i\phi\gamma_5/2}| d\psi = e^{-i\phi\text{Tr}\gamma_5/2} d\psi$

But doesn't $\text{Tr} \gamma_5 = 0$???

Not in the regulated theory!!!

- i.e. $\lim_{\Lambda \rightarrow \infty} \text{Tr} \left(\gamma_5 e^{D^2/\Lambda^2} \right) \neq 0$

Dirac action $\bar{\psi}(D + m)\psi$

- $D^\dagger = -D = \gamma_5 D \gamma_5$

Use eigenstates of D to define $\text{Tr}\gamma_5$

- $D|\psi_i\rangle = \lambda_i|\psi_i\rangle$
- $\text{Tr}\gamma_5 = \sum_i \langle \psi_i | \gamma_5 | \psi_i \rangle e^{-|\lambda_i|^2/\Lambda^2}$

Topology and the index theorem

Require $F_{\mu\nu} \rightarrow 0$ at spatial infinity

- A_μ goes to pure gauge $A_\mu(x) \rightarrow ig^*(x)\partial_\mu g(x)$

For $SU(2)$, $g = a_0 + i\vec{a} \cdot \vec{\sigma}$ with $a_\mu^2 = 1$

- spatial infinity and the group space are both 3-spheres, S_3

$g(x = \infty)$ can wrap non-trivially about the group

- defines a “winding number ‘ ν ’ ” for any gauge field

“Instanton”: an example of a winding number 1 configuration

- minimizes action with winding 1
- a classical solution of the Yang-Mills equations

General $SU(N)$: use $SU(2)$ subgroups

Index theorem

- with gauge winding ν , D has ν zero modes $D|\psi_i\rangle = 0$
 - modes are chiral: $\gamma_5|\psi_i\rangle = \pm|\psi_i\rangle$
 - $\nu = n_+ - n_-$

Non-zero eigenstates in chiral pairs

- $D|\psi\rangle = \lambda|\psi\rangle$
- $D\gamma_5|\psi\rangle = -\lambda\gamma_5|\psi\rangle = \lambda^*\gamma_5|\psi\rangle$

Space spanned by $|\psi\rangle$ and $|\gamma_5\psi\rangle$ gives no contribution to $\text{Tr}\gamma_5$

- $\langle\psi|\gamma_5|\psi\rangle = 0$ when $\lambda \neq 0$
- only the zero modes count!

$$\text{Tr}\gamma_5 = \sum_i \langle\psi_i|\gamma_5|\psi_i\rangle = \nu$$

Where did the opposite chirality states go?

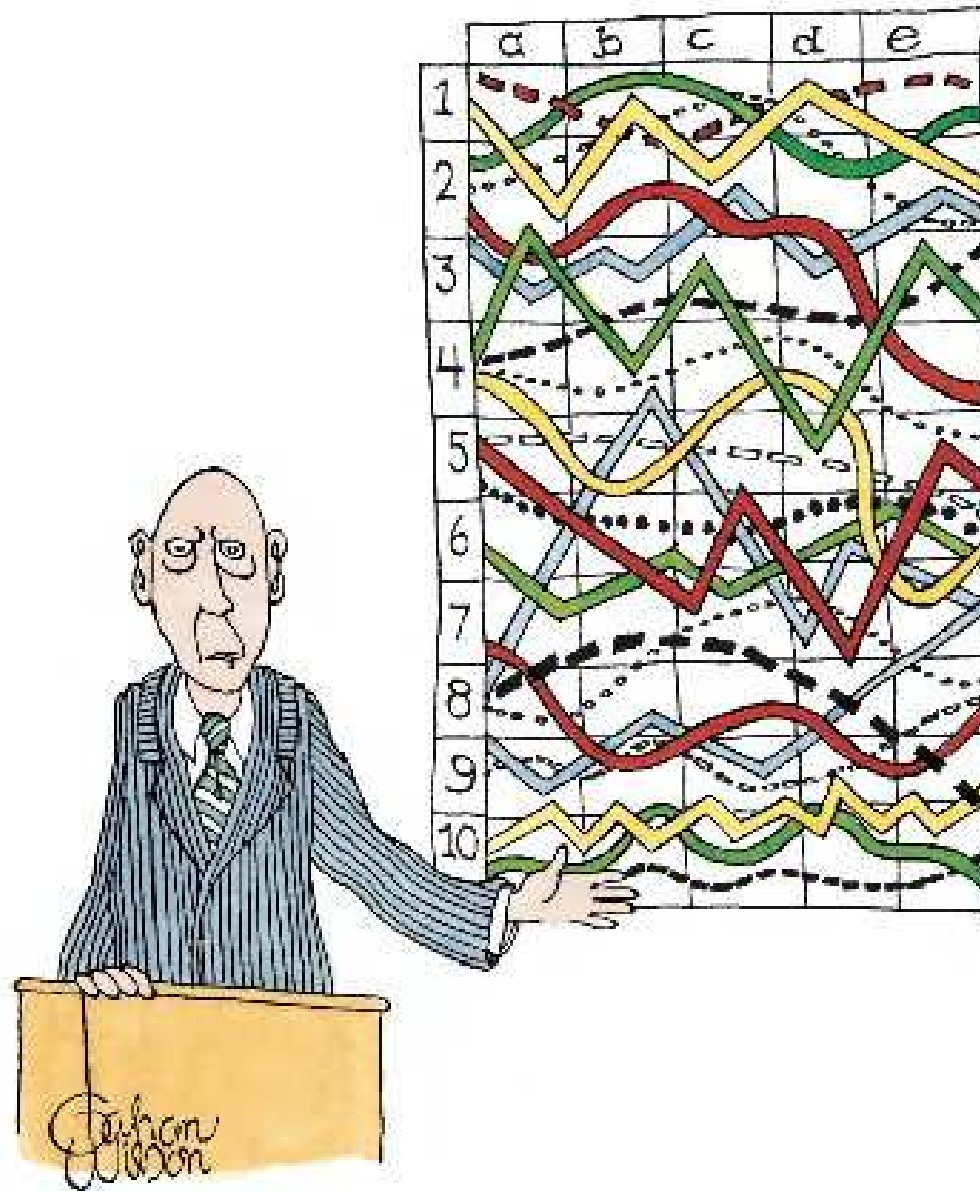
- continuum: lost at “infinity” “above the cutoff”
- Wilson: real eigenvalues in doubler region
- overlap: modes on opposite side of unitarity circle
 - $D\gamma_5 = -\hat{\gamma}_5 D$ $\text{Tr } \hat{\gamma}_5 = 2\nu$

This phenomenon involves both short and long distances

- zero modes compensated by modes lost at the cutoff

Cannot uniquely separate perturbative and non-perturbative effects

- small instantons can “fall through the lattice”
- scheme and scale dependent

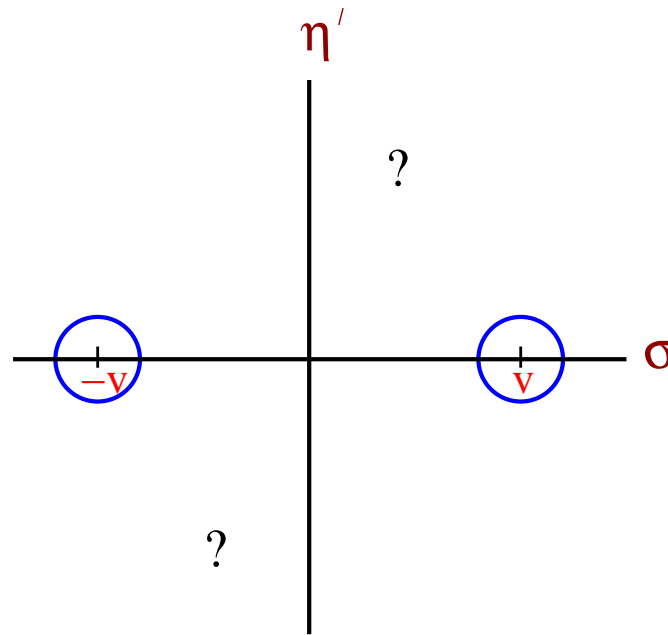


"I'll pause for a moment so you can let this information sink in."

Back to effective field language

At least two minima in the σ, η' plane

$$(\sigma, \eta') = (0, \pm v)$$



Question:

- do we know anything else about the potential in the σ, η' plane?

Yes!

- there are actually N_f equivalent minima

Define $\psi_L = \frac{1+\gamma_5}{2}\psi$ $\psi_R = \frac{1-\gamma_5}{2}\psi$

Singlet rotation of $\psi_L \rightarrow e^{i\phi}\psi_L$ alone

- not a good symmetry for generic ϕ

Flavored rotation $\psi_L \rightarrow g_L\psi_L = e^{i\phi_\alpha\lambda_\alpha}\psi_L$

- is a symmetry for $g_L \in SU(N_f)$

For special discrete values of ϕ these rotations can cross

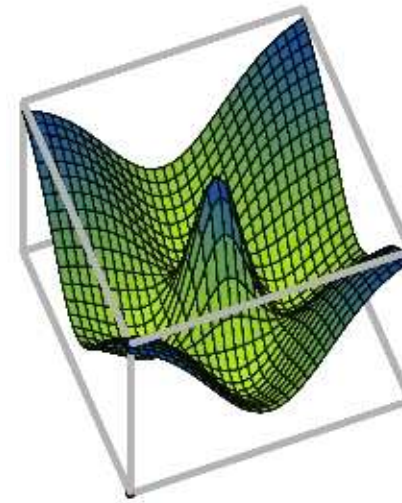
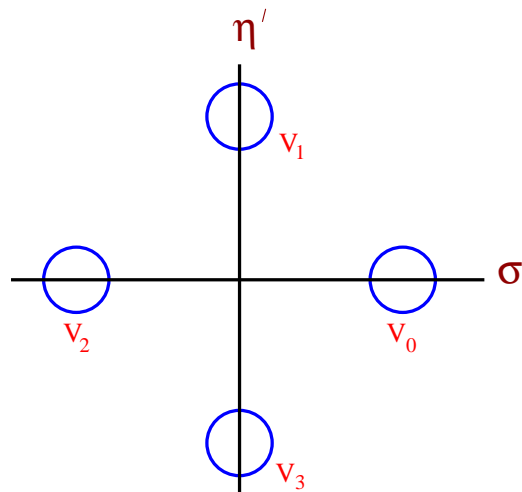
- $g = e^{2\pi i/N_f} \in Z_{N_f} \subset SU(N_f)$

A valid discrete singlet symmetry:

$$\begin{aligned}\sigma &\rightarrow +\sigma \cos(2\pi/N_f) + \eta' \sin(2\pi/N_f) \\ \eta' &\rightarrow -\sigma \sin(2\pi/N_f) + \eta' \cos(2\pi/N_f)\end{aligned}$$

$V(\sigma, \eta')$ has a Z_{N_f} symmetry

- N_f equivalent minima in the (σ, η') plane
- $N_f = 4$:

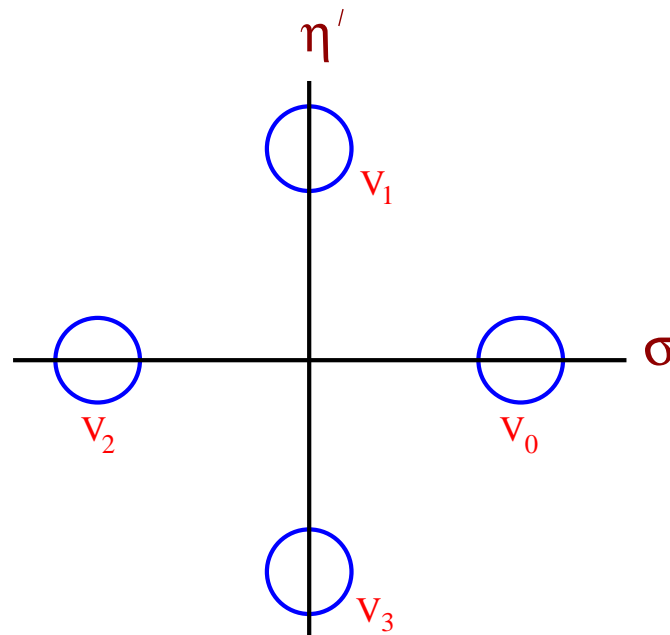


At the chiral lagrangian level

- Z_N is a subgroup of both $SU(N)$ and $U(1)$

At the quark level

- measure gets a contribution from each flavor
- $\psi_L \rightarrow e^{2\pi i/N_f} \psi_L$ is a valid symmetry



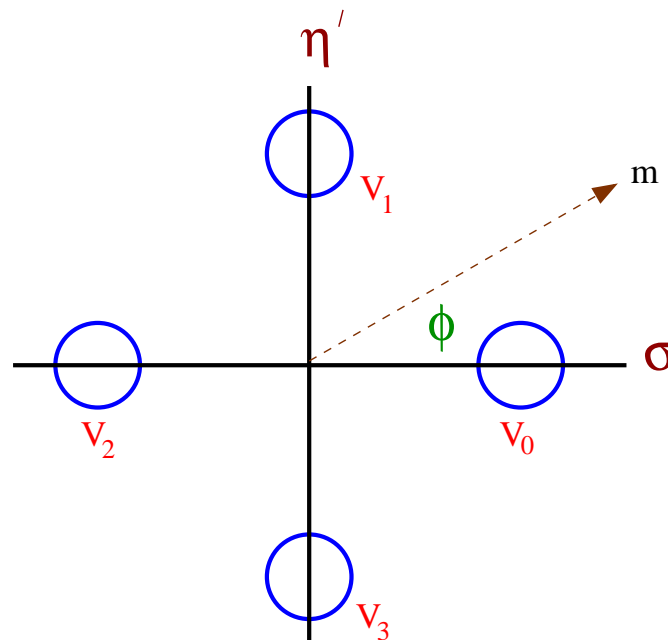
Mass term $m\bar{\psi}\psi$ tilts effective potential

- picks one vacuum (v_0) as the lowest
- in n 'th minimum $m_\pi^2 \sim m \cos(2\pi n/N_f)$
 - highest minima are unstable in the π_α direction
 - multiple meta-stable minima when $N_f > 4$

Anomalous rotation of the mass term

- $m\bar{\psi}\psi \rightarrow m \cos(\phi)\bar{\psi}\psi + im \sin(\phi)\bar{\psi}\gamma_5\psi$
- twists tilt away from the σ direction

An inequivalent theory!

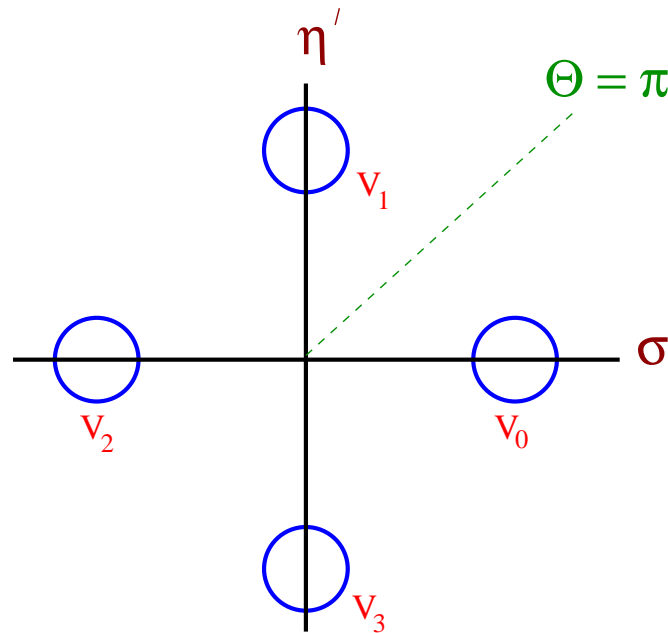


- as ϕ increases, vacuum jumps from one minimum to the next

This is the strong CP parameter Θ

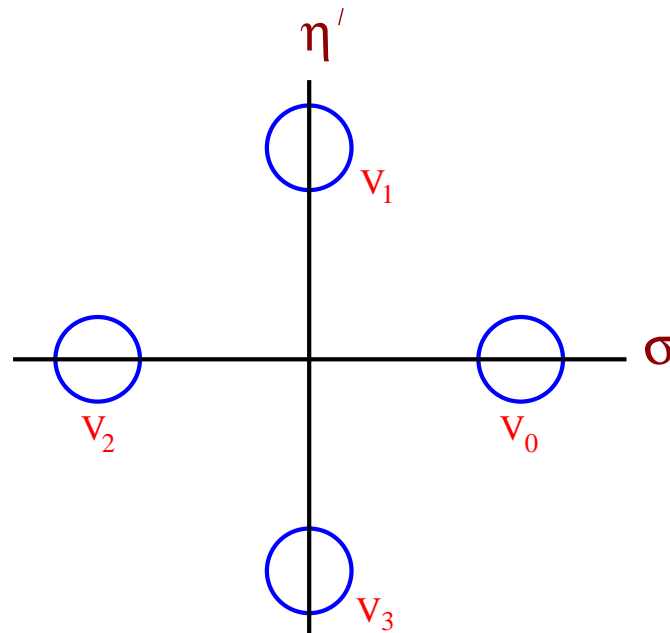
- here each flavor has been given the same phase
- conventional notation uses $\Theta = N_f \phi$
- the Z_{N_f} symmetry implies 2π periodicity in Θ

Degenerate light quarks \Rightarrow first order transition at $\Theta = \pi$



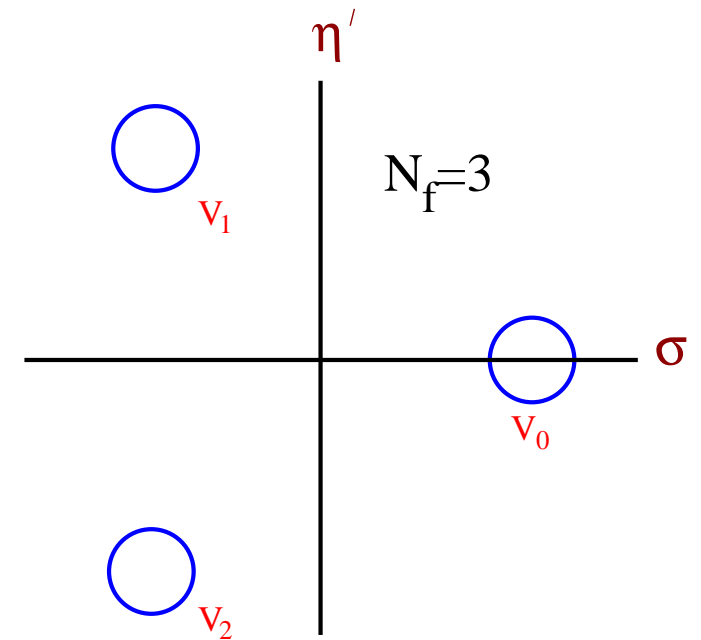
Discrete symmetry in mass parameter space $m \rightarrow m \exp\left(\frac{2\pi i \gamma_5}{N_f}\right)$

- for $N_f = 4$:
 - $m\bar{\psi}\psi$ and $im\bar{\psi}\gamma_5\psi$ mass terms give **equivalent** theories
 - true if and only if N_f is a multiple of 4



Odd number of flavors, $N_f = 2N + 1$

- -1 is not in $SU(2N + 1)$
- $m > 0$ and $m < 0$ not equivalent!
- $m < 0$ represents $\Theta = \pi$
 - an inequivalent theory
 - spontaneous CP violation: $\langle \eta' \rangle \neq 0$

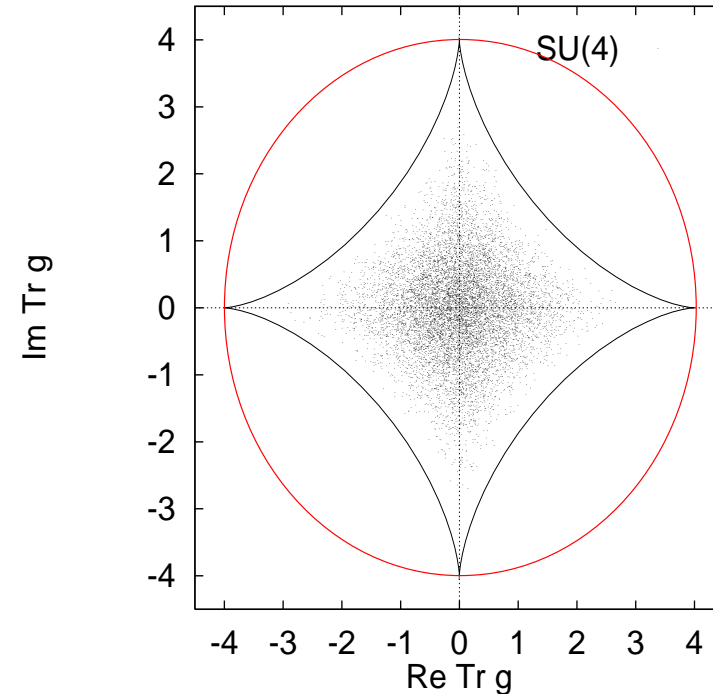
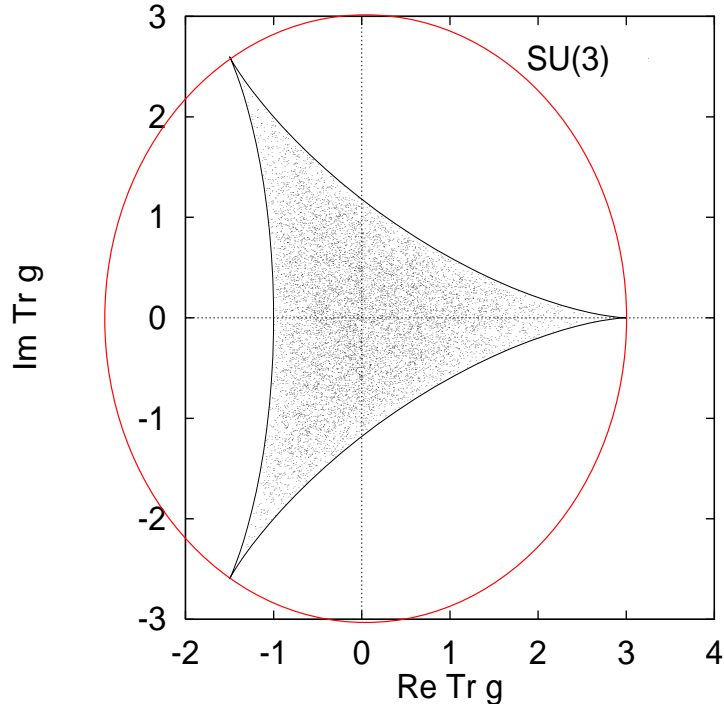


Inequivalent theories can have identical perturbative expansions!

- Theta dependence invisible to perturbation theory

Center of $SU(N_f)$ is a subgroup of $U(1)$

- 10,000 random $SU(3)$ and $SU(4)$ matrices:



- region for $SU(3)$ bounded by $\exp(i\phi\lambda_8)$
- all $SU(N)$ points enclosed by the $U(1)$ circle $e^{i\phi}$
 - boundary reached at center elements

$N_f = 1$: No chiral symmetry at all!

- unique vacuum
- $\langle \bar{\psi}\psi \rangle \sim \langle \sigma \rangle \neq 0$ from 't Hooft vertex
- not a spontaneous symmetry breaking

No singularity at $m = 0$

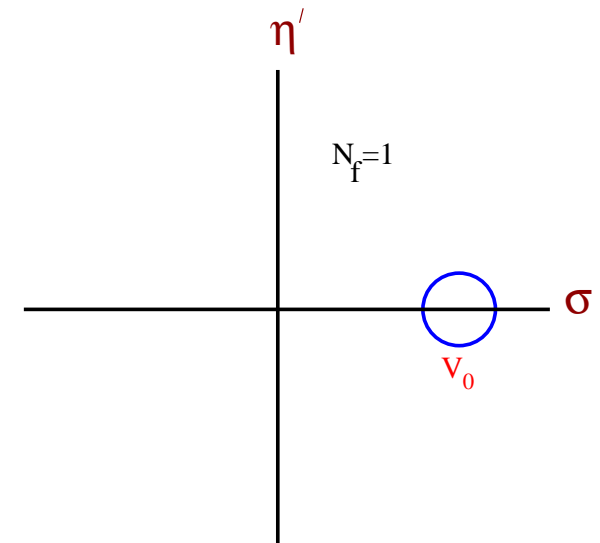
- $m = 0$ not protected: “renormalon” ambiguity

For small mass

- no first order transition at $\Theta = \pi$
 - larger masses?

$N_f = 0$: pure gauge theory

- $\Theta = \pi$ behavior unknown



The rooting question

Starting with four flavors

- can we adjust N_f using

$$\left| \begin{array}{cccc} D + m & 0 & 0 & 0 \\ 0 & D + m & 0 & 0 \\ 0 & 0 & D + m & 0 \\ 0 & 0 & 0 & D + m \end{array} \right|^{\frac{1}{4}} = |D + m|?$$

A vacuous question outside the context of a regulator!

Rooting before removing regulator **OK** if

- regulator **breaks** anomalous symmetries on each factor
 - i.e. four copies of the overlap operator: $D\gamma_5 = -\hat{\gamma}_5 D$
 - winding ν from $\text{Tr}\hat{\gamma}_5 = 2\nu$

Forcing Z_{N_f} symmetry with regulator in place

$$\begin{vmatrix} D + me^{\frac{i\pi\gamma_5}{4}} & 0 & 0 & 0 \\ 0 & D + me^{\frac{-i\pi\gamma_5}{4}} & 0 & 0 \\ 0 & 0 & D + me^{\frac{3i\pi\gamma_5}{4}} & 0 \\ 0 & 0 & 0 & D + me^{\frac{-3i\pi\gamma_5}{4}} \end{vmatrix}$$

- maintains $m \rightarrow me^{i\pi\gamma_5/2}$ symmetry
 - permutes flavors

Four one-flavor theories with different values of Θ

- Theta cancels out for the full four flavor theory

Rooting averages four inequivalent theories: **NOT OK**

- $(|D + M_1||D + M_2|)^{1/2} \neq |D + \sqrt{M_1 M_2}|$

Staggered fermions

- regulator maintains one **exact** chiral symmetry

- $|D_s + m| = |D_s + me^{i\gamma_5\phi}|$

- OK since actually a flavored symmetry

- separate into four “effective tastes” D_i

- two tastes of each chirality

$$|D_s + me^{i\gamma_5\phi}| \sim \begin{vmatrix} D_1 + me^{i\phi\gamma_5} & 0 & 0 & 0 \\ 0 & D_2 + me^{-i\phi\gamma_5} & 0 & 0 \\ 0 & 0 & D_3 + me^{i\phi\gamma_5} & 0 \\ 0 & 0 & 0 & D_4 + me^{-i\phi\gamma_5} \end{vmatrix}$$

- plus “taste mixing” (not the crucial issue)

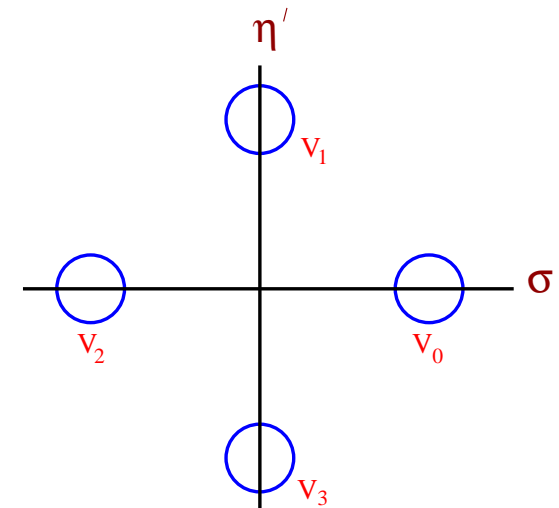
Rooting to get to one flavor **NOT OK**

- rooting does not remove the Z_4
- tastes are not equivalent
 - rooting averages **inequivalent** theories

Rooted staggered fermions are not QCD!

Extra minima from Z_4

- expected to drive η' mass down
 - testable but difficult



Summary

QCD with N_f massless flavors has a discrete Z_{N_f} chiral symmetry

- flavor singlet

Associated with a first order transition at $\Theta = \pi$ when $m \neq 0$

Sign of mass significant for N_f odd

- not seen in perturbation theory

No symmetry for $N_f = 1$

- $m = 0$ unprotected

Structure inconsistent with rooted staggered quarks

Will return to these issues in Lecture IV