Exercises

Lecture 1

1). For $SU(N)$ with the normalization $\int dg 1 = 1$, evaluate the integrals

$$\int dg \text{ Tr } g$$
$$\int dg \ (\text{Tr } g)^2$$
$$\int dg \ (\text{Tr } g)^3$$
$$\int dg \ |\text{Tr } g|^2.$$

2). Show in pure gauge theory that forgetting to integrate over a single arbitrary link leaves all gauge invariant expectations unchanged.

3). In Z_2 lattice gauge theory where all links are plus or minus one, find a configuration where every plaquette is negative.

4). In two dimensional Z_2 pure gauge theory, find an explicit expression for the average plaquette as a function of beta.

5). A Wilson loop is defined as the expectation value for the trace of a product of link variables around a closed loop. Show that in strong coupling for the pure gauge theory these loops fall exponentially with increasing minimal area enclosed by the loop. Should this rapid fall-off persist when quarks are present?

Lecture 2

1). In the SU(2) linear sigma model with degenerate quarks, show that the pion mass is indeed proportional to the square root of the quark mass.

2). Argue that in a gauge theory, changing the sign of the fermion mass in a fermion loop involving more than three interactions leaves the contribution unchanged. How can this argument fail for a triangle diagram.

3). For $g \in SU(N)$ find the locations and values of the maxima and minima of Re Tr g. Are there local maxima or minima that are not global? Are there saddle points?

4). For the three flavor non-linear sigma model, show that the charged and neutral pion mass difference is quadratic in the up-down quark mass difference. How does this work in the two flavor case?

Lecture 3

1). Show that the Wilson Dirac operator is not normal; i.e. D and $D^\dagger$ do not commute. What about the Karsten-Wilczek, overlap, and staggered operators?

2). What is the motivation for using antiperiodic periodic conditions with fermions on a finite lattice?

3). Show that for arbitrary gauge fields the naive fermion matrix can be block diagonalized into four independent factors.

4). Consider a spinless fermions hopping on a lattice with the gauge group $SU(2)$. Show that the theory at negative β is equivalent to conventional staggered fermions.

Lecture 4

1). In the $SU(3)$ non-linear sigma model, show that the Dashen phase starts at

$$m_u = \frac{-m_d m_s}{m_d + m_s}$$

2). The strong CP phase is the phase of the determinant of the mass matrix. Can we rotate Θ into the top quark mass? If we do so, how can it be relevant to low energy physics?

3). The axion solution to the strong CP problem makes the phase of the mass matrix into a dynamical field that relaxes to zero. Show that the anomaly feeds through to give the axion a mass proportional to the trace of the light quark mass matrix. How do the heavy quarks (c,b,t) affect this?

General questions

1). A possible gauge fixing is to set to unity all links on a tree of links containing no loops. Find a tree such that the average expectation of the unfixed links doesn't vanish.

2). There is much discussion of possible zeros of "the" beta function for QCD at non-vanishing coupling. Define the coupling from the force between separated quarks as obtained from Wilson loops. Argue that the beta function associated with this definition of the coupling must have a zero at some $g \neq 0$. Find another definition of the beta function that only has a single zero at the origin.

3). Because of screening by dynamical quark pairs, Wilson loops in QCD with quarks always have a perimeter law. How is confinement defined then?

4). The weak interactions involve the non-Abelian gauge group $SU(2)$. Isn't this a confining theory? How can we have free electrons and W bosons?

5). What does it mean for a particle to be “elementary”?