Partial answers to the exercises

Lecture 1

1). For $SU(N)$ with the normalization $\int dg 1 = 1$, evaluate the integrals

$$\int dg \text{ Tr } g = 0$$

$$\int dg \ (\text{Tr } g)^2 = \delta_{N.2}$$

$$\int dg \ (\text{Tr } g)^3 = \delta_{N,3}$$

$$\int dg \ |\text{Tr } g|^2 = 1.$$

These follow from combining representations. For example, in SU(3) we have $3 \otimes \overline{3} = 1 \oplus 8$ and $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$.

2). Show in pure gauge theory that forgetting to integrate over a single arbitrary link leaves all gauge invariant expectations unchanged.

3). In Z_2 lattice gauge theory where all links are plus or minus one, find a configuration where every plaquette is negative.

A:

$$U_x = 1$$
$$U_y = (-1)^x$$
$$U_z = (-1)^{x+y}$$
$$U_t = (-1)^{x+y+z}$$

4). In two dimensional Z_2 pure gauge theory, find an explicit expression for the average plaquette as a function of beta.

A:

$$Z = \text{Tr} T^N$$

$$T = \begin{pmatrix} e^\beta & e^{-\beta} \\ e^{-\beta} & e^\beta \end{pmatrix}$$

Eigenvalues

$$\lambda = e^\beta \pm e^{-\beta}$$

$$Z = \lambda_+^N + \lambda_-^N \to \lambda_+^N$$

$$P = \frac{1}{N} \frac{d \log(Z)}{d\beta} = \tanh(\beta)$$

5). A Wilson loop is defined as the expectation value for the trace of a product of link variables around a closed loop. Show that in strong coupling for the pure gauge theory these loops fall exponentially with increasing minimal area enclosed by the loop. Should this rapid fall-off persist when quarks are present?

A: Use

$$\int dg g = 0$$

along with

$$\int dg g_{ij} {g^\dagger}_{kl} = \delta_{il}\delta_{jk}/N$$

to find $L \sim \beta^A \sim e^{-A\log(\beta)}$

Lecture 2

1). In the SU(2) linear sigma model with degenerate quarks, show that the pion mass is indeed proportional to the square root of the quark mass.

A:

$$V = (\sigma^2 + \pi^2 - v^2)^2 - m\sigma$$

Minimum at

$$0 = 4\sigma(\sigma^2 + \pi^2 - v^2) - m$$

Pion mass

$$M_\pi^2 \sim \frac{\partial V}{\partial \pi^2} = 2(\sigma^2 + \pi^2 - v^2) \sim m/2\sigma \sim m/2v$$

2). Argue that in a gauge theory, changing the sign of the fermion mass in a fermion loop involving more than three interactions leaves the contribution unchanged. How can this argument fail for a triangle diagram.

3). For $g \in SU(N)$ find the locations and values of the maxima and minima of Re Tr g. Are there local maxima or minima that are not global? Are there saddle points?

A:

$$\frac{d}{d\epsilon}(e^{i\epsilon\cdot\lambda}g + g^\dagger e^{-i\epsilon\cdot\lambda})|_{\epsilon=0} = 0$$

implies $\mathrm{ImTr}\lambda g = \mathrm{const.}$ Diagonalizing g says all eigenvalues of the form $e^{i\theta}$ or $-e^{-i\theta}$. Now to second order in ϵ with diagonal generators, max or min requires all to be one or the other. Thus extrema are elements of Z_N. For $N > 4$ there are multiple local maxima. For a saddle point in $SU(3)$,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thanks to Sidney Coleman for this argument.

4). For the three flavor non-linear sigma model, show that the charged and neutral pion mass difference is quadratic in the up-down quark mass difference. How does this work in the two flavor case?

A: Result must be even in this difference, but in more detail

$$\Sigma = e^{i\pi\cdot\lambda}$$

$$V = \mathrm{ReTr}\Sigma m$$

Expanding gives the meson mass matrix.

$$m = \mathrm{diag}(m_u, m_d, m_s)$$

If $m_u \neq m_d$, $\mathrm{Tr}\lambda_3\lambda_8 m$ gives a π_0, η mixing proportional to $m_u - m_d$. Diagonalizing the meson matrix moves the π_0 down proportional to this mixing squared.

Lecture 3

1). Show that the Wilson Dirac operator is not normal; i.e. D and $D^\dagger$ do not commute. What about the Karsten-Wilczek, overlap, and staggered operators?

2). What is the motivation for using antiperiodic periodic conditions with fermions on a finite lattice?

A: For finite temperature, we want loops around the timelike direction to give a positive contribution.

3). Show that for arbitrary gauge fields the naive fermion matrix can be block diagonalized into four independent factors.

A: In any loop any particular γ_μ appears an even number of times. Thus on any site, any spinor component doesn't know about any of the others.

4). Consider a spinless fermions hopping on a lattice with the gauge group $SU(2)$. Show that the theory at negative β is equivalent to conventional staggered fermions.

A: Absorb the staggered factors into the group integrations.

Lecture 4

1). In the $SU(3)$ non-linear sigma model, show that the Dashen phase starts at

$$m_u = \frac{-m_d m_s}{m_d + m_s}$$

A: The meson mass matrix is

$$\mathcal{M}_{\alpha\beta} \propto \text{ Re Tr } \lambda_\alpha \lambda_\beta M$$

The neutral pion and eta mix

$$\mathcal{M}_{3,8} \propto m_u - m_d$$

Diagonalizing the mass matrix gives

$$m_{\pi_0}^2 \propto \frac{2}{3}\left(m_u + m_d + m_s - \sqrt{m_u^2 + m_d^2 + m_s^2 - m_u m_d - m_u m_s - m_d m_s}\right)$$

The Dashen phase starts where this vanishes.

2). The strong CP phase is the phase of the determinant of the mass matrix. Can we rotate Θ into the top quark mass? If we do so, how can it be relevant to low energy physics?

3). The axion solution to the strong CP problem makes the phase of the mass matrix into a dynamical field that relaxes to zero. Show that the anomaly feeds through to give the axion a mass proportional to the trace of the light quark mass matrix. How do the heavy quarks (c,b,t) affect this?

A: Ignoring lots of factors,

$$L_{qcd} \sim FF + \overline{\psi} D\psi + \text{ReTr} m\Sigma + c\text{Re}|\Sigma|$$

Here

$$\Sigma_{ab} = \overline{\psi}_a \psi_b$$

and m is the mass matrix. Chiral symmetry breaking gives Σ its vacuum expectation value, and fluctuations around this are the meson fields

$$\Sigma \sim \exp(i\pi \cdot \lambda + i\eta')$$

The 't Hooft determinant term gives a mass independent contribution to the eta prime mass.

Add an axion field $a(x)$ to remove the phase of the mass matrix

$$m \to m e^{i\xi a(x)}$$

Here the parameter ξ controls the coupling of the axion; it should be small so the axion won't have yet been discovered. Expanding $V = \text{ReTr} m\Sigma + \text{Re}|\Sigma|$ about vanishing meson fields gives

$$V \sim (\eta' + \xi a)^2 \text{Tr} m + {\eta'}^2$$

This gives a mass matrix mixing the eta prime and axion

$$\begin{pmatrix} \xi^2 \mathrm{Tr} m & \xi \mathrm{Tr} m \\ \xi \mathrm{Tr} m & \mathrm{Tr} m + c \end{pmatrix}$$

This has determinant $\xi^2 c \mathrm{Tr} m$ which should be the product of the physical eta prime and axion masses. The important observation is that a single quark mass vanishing does not give a massless axion.

Heavier quarks factor out since for them $\langle \overline{\psi} \psi \rangle$ is dominated by their mass, not the spontaneous breaking.

General questions

1). A possible gauge fixing is to set to unity all links on a tree of links containing no loops. Find a tree such that the average expectation of the unfixed links doesn't vanish.

2). There is much discussion of possible zeros of "the" beta function for QCD at non-vanishing coupling. Define the coupling from the force between separated quarks as obtained from Wilson loops. Argue that the beta function associated with this definition of the coupling must have a zero at some $g \neq 0$. Find another definition of the beta function that only has a single zero at the origin.

3). Because of screening by dynamical quark pairs, Wilson loops in QCD with quarks always have a perimeter law. How is confinement defined then?

4). The weak interactions involve the non-Abelian gauge group $SU(2)$. Isn't this a confining theory? How can we have free electrons and W bosons?

5). What does it mean for a particle to be "elementary"?