On invariant integration over SU(N)[a]

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We give a graphical algorithm for evaluation of invariant integrals of polynomials in SU(N) group elements. Such integrals occur in strongly coupled lattice gauge theory. The results are expressed in terms of totally antisymmetric tensors and Kronecker delta symbols.

For the strong coupling expansion of lattice gauge theory one requires invariant integrals over polynomials in elements of the fundamental gauge group.[1,2] To explicitly exhibit the invariant measure of a group is in principle straightforward but often in practice a rather tedious task. Beginning with some parametrization, i.e., a generalization of the Euler angles for the rotation group, one studies the group transformation properites of a small volume element in this parameter space. Fortunately, symmetry arguments can determine many integrals, thus sidestepping the explicit construction of the invariant measure. We will show how the symmetry properties of the groups SU(N) give rise to a set of rules for evaluation of the integrals arising in strongly coupled gauge theory. This generalizes to arbitrary N the rules of Ref. 2 for SU(3).

Given any compact Lie group, it is well known that there exists a unique normalized integration measure with the properties

$$\int dg f(g) = \int dg f(g_0 g g_1) = \int dg f(g^{-1}), \quad \int dg = 1, \tag{1}$$

where g is the group element being integrated over, $f(g)$ is an arbitrary function of g, and g_0 and g_1 are arbitrary fixed group elements. In this paper we are interested in the group SU(N); so, g represents an N by N unitary matrix of determinant one. We wish to evaluate integrals of the form

$$I = \int dg\, g_{i_1 j_1} \cdots g_{i_n j_n} g^{-1}_{k_1 l_1} \cdots g^{-1}_{k_m l_m}, \tag{2}$$

where matrix indices for the g's and g^{-1}'s are explicitly indicated. We introduce a generating function for such integrals

$$W(J,K) = \int dg \exp[\mathrm{Tr}(Jg + Kg^{-1})], \tag{3}$$

where J and K are arbitrary N by N matrices. Integrals of the form of Eq. (2) are obtained from $W(J,K)$ by differentiating

$$I = \left(\frac{\partial}{\partial J_{j_1 i_1}} \cdots \frac{\partial}{\partial J_{j_n i_n}} \frac{\partial}{\partial K_{l_1 k_1}} \cdots \frac{\partial}{\partial K_{l_n k_n}} W(J,K) \right) \Bigg|_{J=K=0}. \tag{4}$$

We wish to express $W(J,K)$ in a convenient form that will permit a graphical evaluation of these derivatives.

We first eliminate the K dependence of W by expressing g^{-1} in terms of the cofactor of g. The cofactors of a matrix are easily extracted using the totally antisymmetric tensor $\epsilon_{i_1,\ldots,i_N}$ which satisfies

$$\epsilon_{1,2,\ldots,N} = 1. \tag{5}$$

Since g is of determinant one we obtain the simple expression

$$g^{-1}_{ij} = (\mathrm{cof} g)_{ji} = \frac{1}{(N-1)!} \epsilon_{j,i_1,\ldots,i_{N-1}} \epsilon_{i,j_1,\ldots,j_{N-1}} g_{i_1 j_1}, \ldots, g_{i_{N-1} j_{N-1}}. \tag{6}$$

Using this, multiple derivatives with respect to J can replace derivatives with respect to K; thus, we write

$$W(J,K) = \exp\left\{ \mathrm{Tr}\left[K \left(\mathrm{cof} \frac{\partial}{\partial J} \right) \right] \right\} W(J), \tag{7}$$

where

$$W(J) = \int dg \exp(\mathrm{Tr} Jg). \tag{8}$$

To evaluate $W(J)$ we make use of the invariance of the integration measure, which immediately implies

$$W(J) = W(g_0 J g_1), \tag{9}$$

where g_0 and g_1 are arbitrary matrices in SU(N). In an appendix of Ref. 2 it is proven that any analytic function of J satisfying Eq. (9) is expressible as a power series in the determinant of J. Thus we write

$$W(J) = \sum_{i=0}^{\infty} a_i (\det J)^i. \tag{10}$$

The fact that the integration measure is normalized implies

$$a_0 = 1. \tag{11}$$

We now derive a recursion relation to determine further a_n. Since elements of SU(N) have determinant one, $W(J)$ must satisfy the differential equation

$$\left(\det \frac{\partial}{\partial J} \right) W(J) = W(J). \tag{12}$$

A combinatoric exercise in the Appendix shows

$$\left(\det \frac{\partial}{\partial J} \right) (\det J)^i = \frac{(i+N-1)!}{(i-1)!} (\det J)^{i-1}. \tag{13}$$

From Eqs. (10), (12), and (13) we obtain

$$a_i = \frac{(i-1)!}{(i+N-1)!} a_{i-1}. \tag{14}$$

With Eq. (11) this is solved by

$$a_i = \frac{2!\,3! \cdots (N-1)!}{i!\,(i+1)! \cdots (i+N-1)!}, \tag{15}$$

giving the expression

$$W(J) = \sum_{i=0}^{\infty} \frac{2! \cdots (N-1)!}{i! \cdots (i+N-1)!} (\det J)^i. \tag{16}$$

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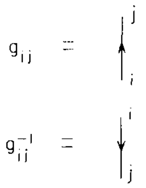

FIG. 1. Graphical representation of g and g^{-1}.

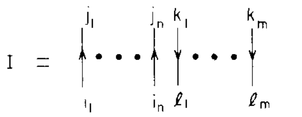

FIG. 2. The generic integral under consideration.

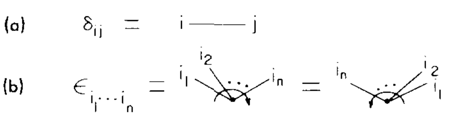

FIG. 3. (a) Representation of the Kronecker symbol; (b) Representation of the antisymmetric tensor.

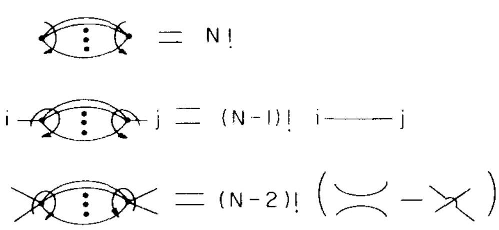

FIG. 4. Some combinatoric identities.

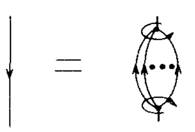

FIG. 5. Replacing g^{-1} with the cofactors of g.

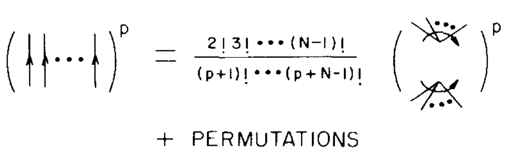

FIG. 6. Evaluation of the integral. There are $(NP)!/[p!(N!)^p]$ distinct permutations to be summed.

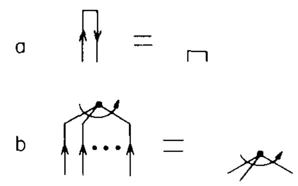

FIG. 7. (a) Invariance of the Kronecker symbol; (b) Invariance of the antisymmetric tensor.

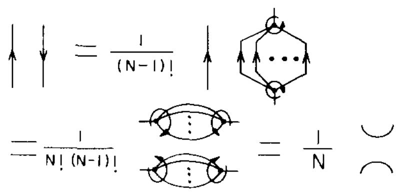

FIG. 8. Evaluation of the integral $\int dg\, g_{ij}\, g^{-1}_{kl}$.

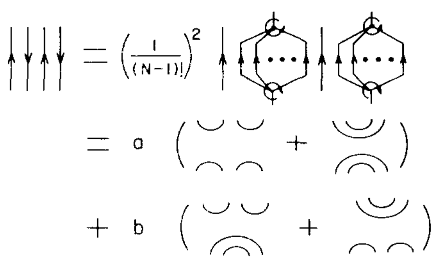

FIG. 9. The integral $\int dg\, g_{i_1j_1}\, g^{-1}_{k_1l_1}\, g_{i_2j_2}\, g^{-1}_{k_2l_2}$.

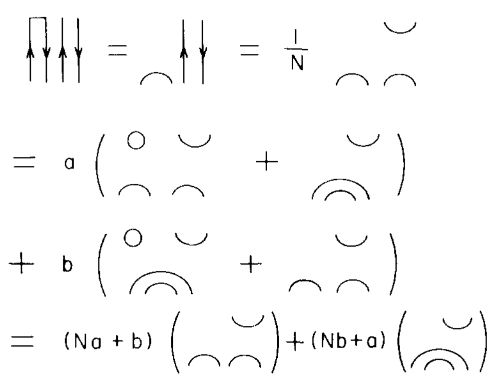

FIG. 10. Evaluation of the coefficients a and b. The closed circles represent $\Sigma_i \delta_{ii} = N$.

Note that the determinant of a matrix is simply expressed in terms of the antisymmetric tensor ϵ

$$\det J = \frac{1}{N!}\, \epsilon_{i_1,\dots,i_N}\, \epsilon_{j_1,\dots,j_N}\, J_{i_1j_1}, \dots, J_{i_Nj_N}. \tag{17}$$

A graphical notation is useful for carrying out the derivatives Eq. (4). Directed vertical line segments correspond to group elements. Upward directed lines represent factors of g while downward directed lines represented factors of g^{-1}, as illustrated in Fig. 1. The ends of these line segments are labeled with the matrix indices of the respective group elements. The line direction runs from the first to the second index, as shown in the figure. Figure 2 shows how the integral of Eq. (2) appears in this notation.

We represent the Kronecker delta symbol δ_{ij} with an undirected line segment connecting the indices i and j, as shown in Fig. 3a. The antisymmetric symbol $\epsilon_{i_1,\dots,i_N}$ is represented by a vertex joining N lines from the indices $i_1, \dots, i_N$. These N lines are labeled with an arrow representing the order of the associated indices in the ϵ symbol, as shown in Fig. 3(b). Finally, when two line segments are connected, a matrix product is understood; i.e., the indices associated with the connected ends are summed over. In the evaluation of group integrals, products of ϵ symbols will often occur. Some useful identities involving such products are:

$$\epsilon_{i_1,\ldots,i_N}\epsilon_{i_1,\ldots,i_N}=N!,$$
$$\epsilon_{i,i_1,\ldots,i_{N-1}}\epsilon_{j,i_1,\ldots,i_{N-1}}=(N-1)!\,\delta_{ij}, \tag{18}$$
$$\epsilon_{i,j,i_1,\ldots,i_{N-2}}\epsilon_{k,l,i_1,\ldots,i_{N-2}}=(N-2)!\,(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}).$$

These have the simple graphical representation shown in Fig. 4.

Evaluation of a group integral consists of a replacement of the directed lines in Fig. 2 with vertices and undirected lines, thus expressing the result in terms of antisymmetric ϵ and Kronecker δ symbols. The first step in this procedure is to convert all directed lines into a set of lines directed upward. This is accomplished using Eq. (6) which is shown graphically in Fig. 5. (If initially there are more downward lines than upward ones it would be equivalent and simpler to convert all lines to downward ones.) Once all lines have the same orientation, we can use Eqs. (16) and (17) to reduce these lines to a sum of terms involving ϵ symbols. Noting that the integral vanishes unless the number of group lines is a multiple of N, Eq. (16) becomes graphically Fig. 6. The indicated sum over permutations is over topologically distinct ways of connecting the group indices to pairs of ϵ vertices and does not include mere permutations of group indices coupled to the same vertex pair or permutations of the vertex pairs. The resulting sum for Np lines has $(Np)!/[p!(N!)^p]$ terms.

Certain identities on the group elements have a simple graphical representation. For example invariance of the Kronecker δ symbol

$$g_{ij}\delta_{jk}g^{-1}_{kl}=\delta_{il}, \tag{19}$$

is shown in Fig. 7(a). Invariance of the ϵ symbol

$$g_{i_1j_1}g_{i_2j_2}\cdots g_{i_Nj_N}\epsilon_{j_1,\ldots,j_N}=\epsilon_{i_1,\ldots,i_N}, \tag{20}$$

is shown in Fig. 7(b). Both of these identities must be true regardless of other lines present in the diagram.

We conclude this paper with some examples of simple integrals to illustrate the rules. First consider $p=1$ in Fig. 6. This immediately gives

$$\int dg\, g_{i_1j_1}\cdots g_{i_Nj_N}=\frac{1}{N!}\,\epsilon_{i_1,\ldots,i_N}\epsilon_{j_1,\ldots,j_N}. \tag{21}$$

Now consider the integral

$$I_{ijkl}=\int dg\, g_{ij}\, g^{-1}_{kl}, \tag{22}$$

shown graphically in Fig. 8. In this figure we use Fig. 5 to make all lines direct upwards, then we use Fig. 6 for $p=1$ to eliminate these lines, and we finally use an identity of Fig. 4 to reduce the result to

$$I_{ijkl}=\frac{1}{N}\,\delta_{jk}\delta_{il}. \tag{23}$$

As a final example consider the integral

$$I=\int dg\,(g_{i_1j_1}g^{-1}_{k_1l_1}g_{i_2j_2}g^{-1}_{k_2l_2}). \tag{24}$$

In Fig. 9 we use Fig. 5 to express I in terms of $2N$ upward lines. Use of Fig. 6 at this point would give an expression with $(2N)!/(2!N!)$ terms; however, this evaluation can be simplified with some tricks. First note that the resulting terms will all have four, an even number, ϵ vertices both at the top and at the bottom of the diagram. These can be eliminated using identities similar to Eq. (18) to reduce the terms to sets of Kronecker δ symbols connecting separately indices at the top and at the bottom of the diagram. Furthermore note that a Kronecker δ cannot connect the indices i_1 and i_2 because they can be initially coupled only through an odd number of ϵ verticles. Using a similar conclusion on the indices j_1 and j_2, we see that the final answer for the integral must take the form

$$I=a(\delta_{i_1l_1}\delta_{i_2l_2}\delta_{j_2k_2}+\delta_{i_1l_2}\delta_{i_2l_1}\delta_{j_1k_2}\delta_{j_2k_1})$$
$$+b(\delta_{i_1l_1}\delta_{i_2l_2}\delta_{j_1k_2}\delta_{j_2k_1}+\delta_{i_1l_2}\delta_{i_2l_1}\delta_{j_1k_1}\delta_{j_2k_2}), \tag{25}$$

where only two independent coefficients are needed because of the $k_1l_1k_2l_2 \leftrightarrow k_2l_2k_1l_1$ symmetry of the integrand. The coefficients a and b can now be determined by multiplying by $\delta_{j_1k_1}$ and using Fig. 7(a) to reduce the integral to that in Fig. 8. This sequence of steps is illustrated in Fig. 10 and leads to the conclusion

$$a=\frac{1}{N^2-1},\quad b=\frac{-1}{N(N^2-1)}. \tag{26}$$

Inserting this in Eq. (25) gives the desired integral.

APPENDIX

Here we prove Eq. (13). Defining

$$f(J)=\left(\det\frac{\partial}{\partial J}\right)(\det J)^i, \tag{A1}$$

we first note that properties of the determinant imply

$$f(J)=f(g_0Jg_1), \tag{A2}$$

for arbitrary g_0 and g_1 in SU(N). By the theorem mentioned below Eq. (9), $f(J)$ must be a function only of $\det J$. By homogeneity we conclude

$$f(J)=C(N,i)(\det J)^{i-1}, \tag{A3}$$

where $C(N,i)$ will now be determined by a recursion relation. Setting $J_{ij}=\delta_{ij}$, we have

$$C(N,i)=\left(\det\frac{\partial}{\partial J}\right)(\det J)^i\Big|_{J_{ij}=\delta_{ij}}. \tag{A4}$$

Writing $\det(\partial/\partial J)$ in terms of ϵ symbols and isolating the sum over minors of the last row gives

$$C(N,i)=\sum_{j=1}^{N}\frac{\partial}{\partial J_{N,j}}\left(\epsilon_{i_1,\ldots,i_{N-1},j}\frac{\partial}{\partial J_{1,i_1}}\cdots\frac{\partial}{\partial J_{N-1,i_{N-1}}}\right)$$
$$\times(\det J)^i\big|_{J_{ij}=\delta_{ij}}. \tag{A5}$$

When $j=N$ in this sum we obtain i times $C(N-1,i)$, while by symmetry all $(N-1)$ terms for $j\neq N$ are equal. Separating the sum over the next to the last row gives

$$C(N,i)$$
$$=iC(N-1,i)+(N-1)\frac{\partial}{\partial J_{N,N-1}}\sum_{j=1}^{N}\frac{\partial}{\partial J_{N-1,j}}$$

$$\times\left(\epsilon_{i_1,\ldots,i_{N-2},j,N-1}\frac{\partial}{\partial J_{1,i_1}}\cdots\frac{\partial}{\partial J_{N-2,i_{N-2}}}\right)(\det J)^i\Big|_{J_{ij}=\delta_{ij}} \tag{A6}$$

In this sum, when $j=N$ we obtain i times $C(N-2,i)$, when $j=N-1$ we get no contribution, and when $j\leq N-2$ we have $N-2$ equal terms. Repeating this process on further rows gives

$$C(N,i)=i\{C(N-1,i)+(N-1)C(N-2,i)+(N-1)(N-2)C(N-3,i)+\cdots(N-1)!\,C(1,i)\}. \tag{A7}$$

Combining (A7) for N and for $N-1$, we see

$$C(N,i)=(i+N-1)\,C(N-1,i). \tag{A8}$$

Using the initial condition $C(1,i)=i$, we conclude

$$C(N,i)=\frac{(i+N-1)!}{(i-1)!}, \tag{A9}$$

which gives Eq. (13).

[1] K. Wilson, Phys. Rev. D **10**, 2445 (1974); L. P. Kadanoff, Rev. Mod. Phys. **49**, 267.

[2] M. Creutz, "Feynman Rules for Lattice Gauge Theory" (to be published in Rev. Mod. Phys.).