PHYSICAL REVIEW D VOLUME 17, NUMBER 10 15 MAY 1978

Higgs mechanism in the temporal gauge

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(Received 15 February 1978)

Working in the temporal gauge $A_0 = 0$, we study canonical quantization of the Higgs model for giving masses to a gauge field. This gauge differs from those discussed previously in that the symmetry of the theory is not spontaneously broken and the Higgs scalar field does not acquire a vacuum expectation value. Rather, the vector meson acquires its mass through a restoration of symmetry of the vacuum under gauge transformations that do not vanish at spatial infinity.

The temporal gauge $A_0=0$ has proved to be a particularly convenient choice for a canonical treatment of nonperturbative tunneling processes in non-Abelian theories[1,2] and for a Hamiltonian formulation of lattice gauge theory.[3] This gauge possesses unphysical degrees of freedom that are frozen out by requiring physical states to be invariant under local time-independent gauge transformations. In a recent paper[2] we studied the transformation properties of the vacuum under those time-independent gauge transformations which do not vanish at spatial infinity. We argued that in any phase possessing massless gauge particles, symmetry under such transformations could be regarded as being spontaneously broken. The gauge mesons then represent the Goldstone bosons associated with this symmetry breaking. In a phase without massless excitations these symmetries should be restored. Indeed, non-Abelian gauge theories with their conjectured quark and gluon confinement should provide an example of this restoration. The Higgs model,[4] however, provides a simpler example of a gauge theory without massless vector mesons; consequently we are led to study this model in the temporal gauge.

The standard discussion of the Higgs mechanism centers on a nonsinglet scalar field acquiring a vacuum expectation value, thus spontaneously breaking the gauge invariance of the vacuum. In other than the temporal gauge, elimination of A_0 as a dependent coordinate results in long-range instantaneous interactions which eliminate massless Goldstone bosons normally associated with spontaneous symmetry breaking. Gauge transformations then shift the theory between physically equivalent Hilbert spaces. However, in the temporal gauge, A_0 is already eliminated and there are no long-range forces. Instead we will show that the remaining gauge freedom is not spontaneously broken and the scalar fields do not acquire a vacuum expectation value. In this gauge the vector mesons acquire mass through a restoration of vacuum symmetry under the above-mentioned transformations that do not vanish at spatial infinity. As expected, the resulting physical spectrum is the same as that obtained in other gauges; merely the associated language changes.

For simplicity we will discuss only the Abelian model with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} + |D_\mu \phi|^2 - \lambda(|\phi|^2 - a^2)^2, \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

and

$$D_\mu \phi = (\partial_\mu - ieA_\mu)\phi. \tag{3}$$

The quantities λ, a, and e are parameters, A_μ is a Hermitian vector field, and ϕ is a complex scalar field. Later we will briefly discuss the introduction of fermions. In the temporal gauge the dynamical variables are ϕ, ϕ^*, and A_i, with i running from 1 to 3. The conjugate momenta are

$$\begin{aligned} \pi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \partial_0 \phi^* = \dot{\phi}^*, \\ \pi^* &= \partial_0 \phi = \dot{\phi}, \end{aligned} \tag{4}$$

and

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = F_{0i} = \dot{A}_i = E_i, \tag{5}$$

where E_i is the electric field. The resulting Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2} E_i^2 + \pi^*\pi + \tfrac{1}{4} F_{ij} F_{ij} + |(\nabla_i + ieA_i)\phi|^2 + \lambda(|\phi|^2 - a^2)^2. \tag{6}$$

We impose canonical commutation relations at equal times

$$\begin{aligned} &[\pi(x), \phi(y)] = [\pi^*(x), \phi^*(y)] = -i\delta^3(\vec{x} - \vec{y}), \\ &[E_i(x), A_j(y)] = -i\delta_{ij}\delta^3(\vec{x} - \vec{y}), \end{aligned} \tag{7}$$

with other combinations vanishing. The equations of motion follow by taking commutators with the Hamiltonian $H=\int d^3x\,\mathcal{H}(x)$:

$$\begin{aligned}&\mathring{\phi}=i[H,\phi]=\pi^*,\quad \dot{\phi}=\pi,\\&\dot{\pi}=(\nabla_i-ieA_i)^2\phi^*-2\lambda\phi^*(|\phi|^2-a^2),\\&\dot{A}_i=E_i,\\&\dot{E}_i=(\nabla^2\delta_{ij}-\nabla_i\nabla_j)A_j-j_i\end{aligned}\tag{8}$$

Here j_i represents the space components of the electromagnetic current

$$j_i=-ie(\phi^*\nabla_i\phi-\nabla_i\phi^*\phi)+e^2A_i|\phi|^2.\tag{9}$$

The charge density is given by

$$j_0=\frac{ie}{2}(\phi^*\pi^*+\pi^*\phi^*-\pi\phi-\phi\pi),\tag{10}$$

where we have symmetrized products of noncommuting operators with the Weyl ordering.[5] The equations of motion imply current conservation

$$\partial_\mu j_\mu=0.\tag{11}$$

Note that Gauss's law $\nabla\cdot E-j_0=0$ does not follow as a Hamiltonian equation of motion. Rather, we find

$$\partial_0(\nabla\circ E-j_0)=i[H,(\nabla\circ E-j_0)]=0.\tag{12}$$

Since $\nabla\cdot E-j_0$ commutes with the Hamiltonian, it can be simultaneously diagonalized and we impose it as a constraint on physical states. Thus we say that a state $|\Psi\rangle$ is physical if

$$(\nabla\cdot E-j_0)|\Psi\rangle=0.\tag{13}$$

At this point we run into a technical difficulty in that $(\nabla\circ E-j_0)$ has a continuous spectrum when the canonical relations of Eq. (7) are realized in a Hilbert space. The eigenstates of an operator with a continuous spectrum are not normalizable and thus our physical states are formally of infinite norm. This norm is, however, a common factor in all physical states and will not appear in gauge-invariant Green's functions.[6] Alternatively one can regularize as in the lattice theory where the spectrum of $\nabla\circ E-j_0$ is discrete.[3]

Fixing $A_0=0$ still leaves open the possibility of time-independent gauge transformations. Indeed, the Hamiltonian is invariant under

$$\begin{aligned}&A_i(x)\to A_i(x)-\frac{1}{e}\nabla_i\Lambda(\vec{x}),\\&\phi(x)\to e^{i\Lambda(\vec{x})}\phi(x),\end{aligned}\tag{14}$$

where $\Lambda(\vec{x})$ is an arbitrary function of $\vec{x}$. Using the commutation relations in Eq. (7), one can show that the unitary operator

$$U=\exp\left[-\frac{i}{e}\int d^3x(E_i\nabla_i+j_0)\Lambda\right]\tag{15}$$

generates this transformation, i.e.,

$$\begin{aligned}&UA_iU^{-1}=A_i-\frac{1}{e}\nabla_i\Lambda,\\&U\phi U^{-1}=e^{i\Lambda}\phi.\end{aligned}\tag{16}$$

We temporarily assume $\Lambda(x)$ goes to zero rapidly enough at spatial infinity so that we can partially integrate the expression in Eq. (15) to obtain

$$U=\exp\left[+\frac{i}{e}\int d^3x(\nabla\circ E-j_0)\Lambda\right].\tag{17}$$

Because Gauss's law generates local gauge transformations, physical states are gauge invariant:

$$U|\Psi\rangle=|\Psi\rangle.\tag{18}$$

This immediately implies that the field ϕ cannot have an expectation value in any physical state because applying a local gauge transformation gives

$$\langle\Psi|\phi|\Psi\rangle=\langle\Psi|U\phi U^{-1}|\Psi\rangle=e^{i\Lambda}\langle\Psi|\phi|\Psi\rangle.\tag{19}$$

Since $\Lambda(\vec{x})$ is arbitrary at any finite point $\vec{x}$, the average of ϕ must be zero.

We now give a perturbative argument that the spectrum of the theory begins with a massive vector meson and a massive scalar meson. Changing to "polar" variables we write

$$\begin{aligned}&\phi(x)=\left(a+\frac{1}{\sqrt{2}}\chi(x)\right)e^{+i\theta(x)},\\&\phi^*(x)=\left(a+\frac{1}{\sqrt{2}}\chi(x)\right)e^{-i\theta(x)},\\&A_i(x)=W_i(x)-\frac{1}{e}\nabla_i\theta(x).\end{aligned}\tag{20}$$

The fields W_i and χ will, respectively, represent spin-1 and spin-0 particles while θ will be eliminated by the imposition of Gauss's law. Under the gauge transformation of Eq. (14) both W_i and χ are invariant while

$$\theta\to\theta+\Lambda.\tag{21}$$

We define momenta conjugate to these new fields by

$$\pi_{W_i}=\dot{W}_i-\frac{1}{e}\nabla_i\dot{\theta}=E_i,\tag{22}$$

$$\pi_\chi=\dot{\chi},\tag{23}$$

$$\pi_\theta=2\left(a+\frac{1}{\sqrt{2}}\chi\right)^2\dot{\theta}+\frac{1}{e}\nabla_i\dot{W}-\frac{1}{e}\nabla^2\dot{\theta}.\tag{24}$$

Working at a fixed time, we impose the commutation relations

$$\begin{aligned}&[E_i(x),W_j(y)]=-i\delta_{ij}\delta^3(\vec{x}-\vec{y}),\\&[\pi_\chi(x),\chi(y)]=[\pi_\theta(x),\theta(y)]=-i\delta^3(\vec{x}-\vec{y}),\end{aligned}\tag{25}$$

with all other combinations from the set

$\{W_i, \chi, \theta, E_i, \pi_\chi, \pi_\theta\}$ commuting. It is straighforward to show that Eq. (25) implies the canonical Eq. (7).

To express π and π^* in terms of the new coordinates requires a little care because $\dot\theta$ and θ do not commute. We find

$$\pi^* = \partial_0\phi = e^{i\theta}\left\{\frac{1}{\sqrt{2}}\,\pi_\chi - \frac{i}{2[a+(1/\sqrt{2}\,)\chi]}\left(\pi_\theta - \frac{1}{e}\,\nabla\cdot E\right) - \tfrac{1}{2}[\dot\theta,\theta]\left(a+\frac{1}{\sqrt{2}}\,\chi\right)\right\}. \tag{26}$$

Unfortunately the commutator $[\dot\theta,\theta]$ diverges because both fields are taken at the same point. This divergence will later serve to cancel divergent Feynman diagrams coming from the derivative couplings in the Hamiltonian when expressed in these new variables. At this point we abandon rigor and write formally

$$[\dot\theta,\theta] = \left[\frac{1}{2[a+(1/\sqrt{2}\,)\chi]^2}\left(\pi_\theta - \frac{1}{e}\,\nabla\cdot E\right),\theta\right] = -\frac{i}{2[a+(1/\sqrt{2}\,)\chi]^2}\,\delta^3(0)\,, \tag{27}$$

where $\delta^3(0)$ represents a cubic divergence in an ultraviolet cutoff imposed for calculational purposes. Thus we write

$$\pi^* = \tfrac{1}{2}e^{i\theta}\left\{\pi_\chi - \frac{i[\pi_\theta - (1/e)\nabla\cdot E]}{a+(1/\sqrt{2}\,)\chi} + \frac{i\delta^3(0)}{a+(1/\sqrt{2}\,)\chi}\right\}. \tag{28}$$

Using this we can express j_0 in the new coordinates. A little algebra gives the simple result

$$j_0 = -2e\left(a+\frac{1}{\sqrt{2}}\,\chi\right)^2\dot\theta\,. \tag{29}$$

In Eq. (24) this gives

$$\pi_\theta = \frac{1}{e}\,(\nabla\cdot E - j_0)\,. \tag{30}$$

Thus we see that the imposition of Gauss's law is the same as setting

$$\pi_\theta|\Psi\rangle = 0 \tag{31}$$

for physical states. Note also that Eq (21) shows that π_θ is the generator of time-independent gauge transformations.[7]

Using Eqs. (28), (20), and (6), we express the Hamiltonian density in the new variables

$$\mathcal{H} = \tfrac{1}{2}E_i^{\,2} + \tfrac{1}{2}\pi_\chi^{\,2} + \frac{1}{4[a+(1/\sqrt{2}\,)\chi]^2}\left[\left(\pi_\theta - \frac{1}{e}\,\nabla\cdot E\right)^2 - \frac{[\delta^3(0)]^2}{4}\right] + \tfrac{1}{2}(\nabla_i\chi)^2 + e^2\left(a+\frac{1}{\sqrt{2}}\,\chi\right)^2 W_i^{\,2} + \tfrac{1}{4}F_{ij}F_{ij} + \lambda\left[\left(a+\frac{1}{\sqrt{2}}\,\chi\right)^2 - a^2\right]^2. \tag{32}$$

The term proportional to $[\delta^3(0)]^2$ is a field-theoretical generalization of the quantum corrections to the centrifugal barrier obtained in going to polar coordinates in a cylindrically symmetric ordinary quantum-mechanics problem.[8]

We wish to set up a perturbation theory in e with the masses of the W_i and χ particles fixed. To that end we define

$$m_\chi^{\,2} = 4a^2,\qquad m_W^{\,2} = 2e^2a^2. \tag{33}$$

Restricting ourselves to the physical Hilbert space, we set π_θ in Eq. (32) to zero and eliminate λ and a with Eq. (33). The Hamiltonian then splits naturally into two parts

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I\,, \tag{34}$$

where

$$\mathcal{H}_0 = \tfrac{1}{2}\pi_\chi^{\,2} + \tfrac{1}{2}(\nabla_i\chi)^2 + \tfrac{1}{2}m_\chi^{\,2}\chi^2 + \tfrac{1}{2}E_i^{\,2} + \frac{1}{2m_W^{\,2}}(\nabla\cdot E)^2 + \tfrac{1}{2}W_i(-\delta_{ij}\nabla^2 + \nabla_i\nabla_j)W_j + \tfrac{1}{2}m_W^{\,2}W_i^{\,2} \tag{35}$$

and

$$\mathcal{H}_I = -\left(\frac{e^2\chi^2 + 2e\chi m_W}{2m_W^{\,2}(m_W + e\chi)^2}\right)\left((\nabla\cdot E)^2 - \frac{e^2[\delta^3(0)]^2}{4}\right) + \tfrac{1}{2}(2e\chi m_W + e^2\chi^2)W_i^{\,2} + \frac{em_\chi^{\,2}}{2m_W}\,\chi^3 + \frac{e^2m_\chi^{\,2}}{8m_W^{\,2}}\,\chi^4\,. \tag{36}$$

The bare theory defined by $\mathcal{H}_0$ alone can be exactly solved and then $\mathcal{H}_I$, which is of order e and higher, can be treated as a perturbation. The χ-dependent part of $\mathcal{H}_0$ is clearly the standard Hamiltonian density for a self-conjugate scalar field of mass m_χ and needs no further discussion. The W_i field is less standard so we treat it in some detail. The equations of motion for the bare W_i field are

$$\dot{W}_i = \left(\delta_{ij} - \frac{\nabla_i \nabla_j}{m_W^2}\right) E_j , \tag{37}$$

$$\dot{E}_i = -\left[\left(m_W^2 - \nabla^2\right)\delta_{ij} + \nabla_i \nabla_j\right] W_j . \tag{38}$$

Combining Eqs. (37) and (38) gives

$$\ddot{W}_i = -(m_W^2 - \nabla^2) W_i ;$$

consequently, the bare field W_i satisfies the Klein-Gordon equation with mass m_W. The solution to the theory defined by $\mathcal{H}_0$ is obtained by writing

$$W_i(x_\mu) = \int \frac{d^3k}{2k_0(2\pi)^3} \left[\epsilon_{is}(\vec{k}) a_s(\vec{k}) e^{-ik_\mu x_\mu} + \epsilon_{is}^*(\vec{k}) a_s^\dagger(\vec{k}) e^{+ik_\mu x_\mu}\right] \tag{40}$$

and

$$E_i(x_\mu) = \int \frac{d^3k}{2k_0(2\pi)^3} (-ik_0)\left(\delta_{ij} - \frac{k_i k_j}{k_0^2}\right) \times \left[\epsilon_{js}(\vec{k}) a_s(\vec{k}) e^{-ik_\mu x_\mu} - \epsilon_{js}^*(\vec{k}) a_s^\dagger(\vec{k}) e^{+ik_\mu x_\mu}\right]. \tag{41}$$

Here $a_s(k)$ are destruction operators for the quanta of the W_i field and satisfy (s runs from one to three)

$$[a_s(\vec{k}), a_{s'}^\dagger(\vec{k}')] = 2k_0(2\pi)^3 \delta^3(\vec{k}' - \vec{k}) \delta_{ss'} . \tag{42}$$

The polarization vectors ϵ_{is} for $s = 1$ and 2 are two arbitrary unit vectors orthogonal to each other and to $\vec{k}$ while

$$\epsilon_{i3} = \hat{k}_i \left(\frac{k_0}{m_W}\right) . \tag{43}$$

The bare vacuum is annihilated by $a_s(\vec{k})$:

$$a_s(\vec{k}) |0\rangle_0 = 0. \tag{44}$$

The subscript 0 on the state means that it is the vacuum of $\mathcal{H}_0$ and not the full $\mathcal{H}$. The bare propagators for the W_i field are

$${}_0\langle 0 | T(W_i(x) W_j(y)) | 0\rangle_0 = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \times \left(\delta_{ij} + \frac{k_i k_j}{m_W^2}\right) , \tag{45}$$

$${}_0\langle 0 | T(E_i(x) W_j(y)) | 0\rangle_0 = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + ie} (-ik_0 \delta_{ij}) , \tag{46}$$

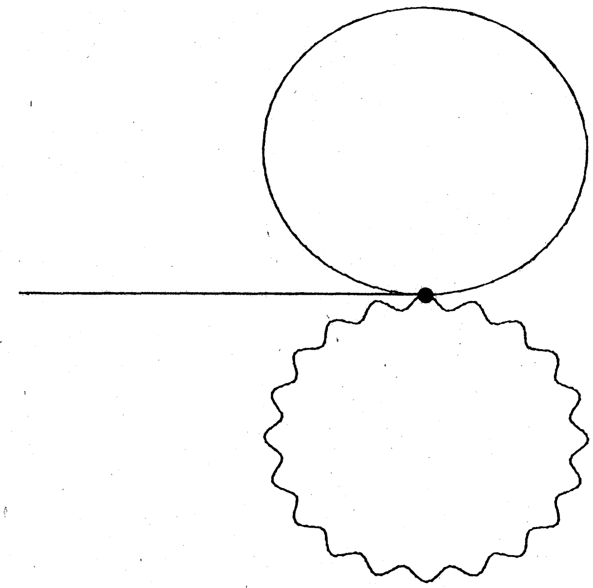

FIG. 1. A diagram contributing to the χ one-point Green's function. The solid and wavy lines represent scalar and vector propagators, respectively.

$${}_0\langle 0 | T(E_i(x) E_j(y)) | 0\rangle_0 = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \times \left[(\vec{k}^2 + m^2)\delta_{ij} - k_i k_j\right]. \tag{47}$$

From these propagators, perturbation theory in $\mathcal{H}_I$ can be worked out in the standard manner. Note that in order e^3 the term involving $[\delta^3(0)]^2$ will begin to appear. This will cancel divergences in diagrams containing both W and χ loops. For example, in Fig. 1 we show an example of a diagram contributing to the one-point Green's function for χ. This diagram is one of several to this order of e that diverge as the sixth power of the cutoff in momentum space. The sum of these divergences should cancel the term in $\mathcal{H}_I$ linear in χ and $[\delta^3(0)]^2$.

The above discussion shows that at least in perturbation theory the physical states of the model are the same as in more usual discussions where the gauge symmetry is spontaneously broken. In terms of the new polar variables it is exactly π_θ that generates gauge transformations. Imposing that π_θ vanish on all physical states is equivalent to saying that physical states are invariant under all gauge transformations. This invariance holds regardless of the asymptotic behavior of the gauge transformation. This model has actually restored symmetry under transformations that do not vanish at spatial infinity, obviating the need for massless gauge mesons as discussed in Ref. 2.

Often the Higgs mechanism is also used to generate masses for fermion fields coupled in a Yukawa manner to the scalar Higgs fields. To see this mass generation in the temporal gauge, where the Higgs fields do not acquire a vacuum expectation value, requires the removal of a gauge-dependent phase from the fermion field. For example, consider a Fermi field that transforms under the gauge transformation of Eq. (14) as

$$\Psi \rightarrow \Psi e^{i\Lambda(x)}. \tag{48}$$

Upon going to polar variables as in Eq. (20), one should also define a new Fermi field

$$\Psi(x) = \Psi'(x) e^{+i\theta(x)}. \tag{49}$$

The new field $\Psi'(x)$ is invariant under gauge transformations, and mass generation will then proceed as in conventional discussions in other gauges.

This work was performed under Contract No. EY-76-C-02-0016 with the U. S. Department of Energy.

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