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Equal-Time Commutators of the Electromagnetic Current and Its Time Derivatives*

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We consider equal-time commutators among various components of the electromagnetic current and its time derivatives. In particular we study the general structure of the vacuum and one-particle matrix elements of these commutators. The form of the commutation relations with no time derivatives enables us to obtain new relations on the commutators involving time derivatives of the current. Using these results, we discuss possible q-number Schwinger terms in the equal-time commutator.

I. INTRODUCTION

Equal-time commutation relations of local operators play an essential role in elementary-particle theory. The canonical equal-time commutation relations of fields are basic to Lagrangian field theory. Hypothesized equal-time commutators for components of observable currents are the starting point for many current-algebra theories. Furthermore, commutators and their time derivatives at equal times are closely related to the asymptotic behavior in energy of particular covariant amplitudes.[1] Recent work in perturbation theory shows that the equal-time commutators given by these relations can be more complicated than expected from naive application of the usual canonical commutators.[2] Nevertheless, causality requires a certain simplicity in the equal-time commutators, which must vanish for spacelike separations.

In this paper we exploit an observation of Furlan and Rossetti and of Cornwall and Norton[3] that the knowledge, through hypothesis or otherwise, of an equal-time commutator in all Lorentz frames places constraints on the time derivatives of the commutator taken at equal times. The most singular parts of a matrix element of time derivatives of a commutator taken at equal times are determined uniquely by the equal-time commutator itself. Less singular terms depend on further dynamics.

We use this technique on the commutator of two electromagnetic currents. We choose to look at electromagnetic currents because these commutators are directly related to hadron production in electromagnetic processes. For a particular matrix element we find the most general form for the commutator which is allowed under general symmetry requirements and with the additional assumption that it is no more singular than a single derivative of a spatial δ function. We make this assumption for simplicity; it is not necessary in general. With this form for the commutator at equal times, we proceed to determine the most singular parts of a time derivative of the commutator taken at equal times. This assumes that the commutator and its time derivative are well defined at equal times.[4] Using our results, we discuss the possibility that the terms involving derivatives of spatial δ functions occurring in an expectation value of the equal-time commutator depend nontrivially on the state with respect to which the expectation value is taken. Such terms are commonly called q-number Schwinger terms. We find a connection between such terms and the experimental result that the total transverse cross section for photoproduction of hadrons on a hadronic target with off-mass-shell photons goes to a constant as the photon energy becomes large with fixed virtual-photon mass.

In Sec. II we consider the vacuum expectation value of the commutator of two electromagnetic currents. Here we can check our results because the matrix element for all times is completely determined by the spectral representation in terms of a single spectral function. Therefore the commutator and all its derivatives at equal times can be calculated explicitly from this spectral function. In this section we also consider briefly the possibility that this matrix element of the commutator is not well defined at equal times and the relation of this to scale invariance. In Sec. III we discuss the single-particle diagonal matrix element of the same commutator. Here we discuss q-number Schwinger terms. We conclude in Sec. IV with a summary of our results.

II. THE VACUUM EXPECTATION VALUE

In this section we consider the vacuum expectation value of the commutator of two electromagnetic currents given by

$$h_{\mu\nu}(x) = \langle 0 | [j_\mu(x), j_\nu(0)] | 0 \rangle . \tag{1}$$

Here $j_\mu(x)$ is the electromagnetic current with a factor of e, the electric charge, removed. Before proceeding let us review the spectral representation of $h_{\mu\nu}(x)$. For the Fourier transform of $h_{\mu\nu}(x)$ we have a simple expression in terms of a single spectral function $\rho(q^2)$ defined by

$$W_{\mu\nu}(q)=\int d^4x\, e^{iq\cdot x}h_{\mu\nu}(x)$$
$$=-\epsilon(q_0)\rho(q^2)\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right), \tag{2}$$

where $\epsilon(q_0)=q_0/|q_0|$. Here $\rho(q^2)$ has the properties

$$\rho(q^2)\geqslant 0,$$
$$\rho(q^2)=0 \quad \text{for} \quad q^2<0. \tag{3}$$

Current conservation restricts us to a single spectral function instead of two. In the remainder of this paper we will only consider the coupling of the electromagnetic current to hadrons, and this only to the lowest nonvanishing order in the electromagnetic charge. Since the lightest intermediate state which can then contribute is the two-pion state, we have

$$\rho(q^2)=0 \quad \text{for} \quad q^2<4m_\pi^{\,2}. \tag{4}$$

The total cross section for e^+e^- annihilation into hadrons at center-of-mass energy $\sqrt{s}$ and to second order in the electromagnetic charge e is directly related to $\rho(q^2)$ by (neglecting the electron mass)

$$\sigma_{e^+e^-\to\text{hadrons}}(\sqrt{s})=\frac{e^4}{2s^2}\rho(s). \tag{5}$$

Because experiment[5] indicates a nonvanishing cross section, $\rho(s)$ cannot vanish identically.

Equation (2) can be Fourier transformed to give

$$h_{\mu\nu}(x)=\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}W_{\mu\nu}(q)$$
$$=i\int_{4m_\pi^2}\frac{d\sigma\,\rho(\sigma)}{2\pi\sigma}\left[(g_{\mu\nu}\Box-\partial_\mu\partial_\nu)\Delta(\sigma,x)\right], \tag{6}$$

where $\Delta(\sigma,x)$ is defined by

$$\Delta(\sigma,x)=-i\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\epsilon(q_0)2\pi\delta(q^2-\sigma) \tag{7}$$

and $\Box=\partial_\mu\partial^\mu$. Equation (6) is the spectral representation for $h_{\mu\nu}(x)$ in terms of $\rho(\sigma)$. From the definition it is readily verified that

$$\begin{aligned}&\delta(x_0)\partial_0^{\,2m}\Delta(\sigma,x)=0,\\&\delta(x_0)\partial_0^{\,2m+1}\Delta(\sigma,x)=(-1)^{m+1}(\sigma+\Box-\partial_0^{\,2})^m\delta^4(x)\\&\qquad=(-1)^{m+1}(\sigma-\vec{\nabla}^2)^m\delta^4(x),\end{aligned} \tag{8}$$

where m is any non-negative integer. Combining (8) with (6) at equal times gives the form of the equal-time commutator

$$\delta(x_0)h_{\mu\nu}(x)=+iC(g_{\mu0}\partial_\nu+g_{\nu0}\partial_\mu-2g_{\mu0}g_{\nu0}\partial_0)\delta^4(x), \tag{9}$$

where C is given by

$$C=\int_{4m_\pi^2}^{\infty}\frac{d\sigma\,\rho(\sigma)}{2\pi\sigma}. \tag{10}$$

We will later discuss briefly the possibility that this integral may be divergent.

With these preliminaries out of the way we can now proceed to discuss the relation of the time derivatives of the commutator at equal times to the equal-time commutator itself. To do this we note that since Eq. (9) is true in all Lorentz frames we can introduce an arbitrary timelike vector n_α with $n_0>0$ and write

$$\delta(n\cdot x)h_{\mu\nu}(x)=\frac{+iC}{\sqrt{n^2}}\left(\frac{n_\mu}{\sqrt{n^2}}\partial_\nu+\frac{n_\nu}{\sqrt{n^2}}\partial_\mu-2\frac{n_\mu n_\nu\partial\cdot n}{(\sqrt{n^2})^3}\right)\delta^4(x). \tag{11}$$

This equation should be true for all n satisfying $n^2>0$, $n_0>0$; so we can differentiate with respect to n_λ to obtain

$$\begin{aligned}n^2\frac{d}{dn^\lambda}\delta(n\cdot x)h_{\mu\nu}(x)&=x_\lambda\delta'\left(\frac{n\cdot x}{\sqrt{n^2}}\right)h_{\mu\nu}(x)\\&=iC\Bigg[\left(g_{\mu\lambda}-\frac{2n_\mu n_\lambda}{n^2}\right)\partial_\nu+\left(g_{\nu\lambda}-\frac{2n_\nu n_\lambda}{n^2}\right)\partial_\mu\\&\qquad-2\left(\frac{g_{\mu\lambda}n_\nu\partial\cdot n}{n^2}+\frac{g_{\nu\lambda}n_\mu\partial\cdot n}{n^2}+\frac{n_\mu n_\nu n_\lambda}{n^2}-4\frac{n_\mu n_\nu n_\lambda\partial\cdot n}{n^4}\right)\Bigg]\delta^4(x).\end{aligned} \tag{12}$$

Equation (12) can be solved to find the most general form for the generalized function $\delta'(n\cdot x/\sqrt{n^2})h_{\mu\nu}(x)$. This gives

$$\delta'\left(\frac{n\cdot x}{\sqrt{n^2}}\right)h_{\mu\nu}(x)=\left[-iC\left(\partial_\mu\partial_\nu-\frac{2n_\mu n\cdot\partial\partial_\nu}{n^2}-\frac{2n_\nu n\cdot\partial\partial_\mu}{n^2}-\frac{n_\mu n_\nu}{n^2}\Box+4\frac{n_\mu n_\nu(\partial\cdot n)^2}{n^4}\right)-iK_{\mu\nu}\right]\delta^4(x). \tag{13}$$

Here $K_{\mu\nu}$ is a Lorentz tensor undetermined by Eq. (12). It depends only on $n_\alpha/\sqrt{n^2}$ and contains no derivatives with respect to x. Knowledge of $\delta'(n\cdot x)/\sqrt{n^2})h_{\mu\nu}(x)$ implies knowledge of $\delta(n\cdot x/\sqrt{n^2})(\partial\cdot n/\sqrt{n^2})h_{\mu\nu}(x)$ through the identity

$$\delta\left(\frac{n\cdot x}{\sqrt{n^2}}\right)\frac{\partial\cdot n}{\sqrt{n^2}}h_{\mu\nu}(x)=\frac{\partial\cdot n}{\sqrt{n^2}}\left[\delta\left(\frac{n\cdot x}{\sqrt{n^2}}\right)h_{\mu\nu}(x)\right]-\delta'\left(\frac{n\cdot x}{\sqrt{n^2}}\right)h_{\mu\nu}(x). \tag{14}$$

Using Eqs. (11), (13), and (14) gives the result

$$\delta\left(\frac{n\cdot x}{\sqrt{n^2}}\right)\frac{\partial\cdot n}{\sqrt{n^2}}h_{\mu\nu}(x)=iC\left(\partial_\mu\partial_\nu-\frac{n_\mu n_\nu}{n^2}\Box+2\frac{n_\mu n_\nu(\partial\cdot n)^2}{n^4}-\frac{n_\mu n\cdot\partial\partial_\nu}{n^2}-\frac{n_\nu n\cdot\partial\partial_\mu}{n^2}\right)\delta^4(x)+iK_{\mu\nu}\delta^4(x). \tag{15}$$

Current conservation gives us some additional information on $K_{\mu\nu}$. Since $\partial_\mu h_{\mu\nu}=0$ we have

$$\partial_\mu\delta(n\cdot x)h_{\mu\nu}(x)=n_\mu\delta'(n\cdot x)h_{\mu\nu}(x). \tag{16}$$

This can agree with Eq. (13) only if we require $n_\mu K_{\mu\nu}=0$. By symmetry we have $n_\nu K_{\mu\nu}=0$. This means that in general we must have

$$K_{\mu\nu}\left(\frac{n}{\sqrt{n^2}}\right)=\left(g_{\mu\nu}-\frac{n_\mu n_\nu}{n^2}\right)D, \tag{17}$$

where D is a dynamical constant which may be infinite. Writing our result in terms of specific components and setting $n_\lambda=g_{\lambda 0}$ gives

$$\delta(x_0)\partial_0 h_{00}(x)=iC(-\Box+\partial_0{}^2)\delta^4(x)=iC\vec{\nabla}^2\delta^4(x), \tag{18a}$$

$$\delta(x_0)\partial_0 h_{0i}(x)=0, \tag{18b}$$

$$\delta(x_0)\partial_0 h_{ij}(x)=iC\partial_i\partial_j\delta^4(x)+iDg_{ij}\delta^4(x). \tag{18c}$$

Here i and j run from 1 to 3.

Using only information about the commutator at equal times, we have found the general form for the time derivative of the commutator at equal times. Clearly this process could be repeated to find higher time derivatives of $h_{\mu\nu}(x)$ at $x_0=0$. At each stage another dynamical parameter will be introduced. If this parameter is infinite, the corresponding commutator and higher derivatives are not well defined at equal times. Always the most singular part will be determined by the equal-time commutator alone.

Since Eq. (6) gives the commutator explicitly for all times in terms of $\rho(\sigma)$, one can differentiate with respect to time and go to equal times to check our result. This procedure immediately verifies our conclusions and gives the additional result that

$$D=\int_{4m^2}^{\infty}\frac{d\sigma}{2\pi}\rho(\sigma). \tag{19}$$

Clearly the additional constants obtained by looking at higher derivatives are related to higher-moment integrals of $\rho(\sigma)$. Unless $\rho(\sigma)$ goes to zero faster than any power as σ goes to infinity, eventually one of these moment integrals will diverge. This means that probably not all time derivatives of the commutator are well defined at equal times. For example, if D is a divergent quantity, the interpretation of Eq. (18c) is unclear.

For the remainder of this section we would like to digress and discuss how the spectral representation can give interesting results at equal times even when the commutator itself is undefined there. This occurs when the integral defining C in Eq. (10) diverges. These results are not new[6] but we discuss them in a slightly different language. Let us assume, for example, that $\rho(\sigma)/\sigma$ goes to a constant as σ goes to infinity. We also assume this limit is sufficiently nonpathological that we can use the l'Hospital relation $\rho(\sigma)/\sigma \sim (d/d\sigma)\rho(\sigma)$ as $\sigma\to\infty$. Although $h_{\mu\nu}(x)$ is now undefined at $x_0=0$, the spectral representation still completely determines the commutator for unequal times. Thus we can consider x^2 times the commutator and see if this has a well-defined equal-time limit. Using the identities

$$\begin{aligned}x_\mu\Delta(\sigma,x)&=-2\partial_\mu\frac{d}{d\sigma}\Delta(\sigma,x),\\ x^2\Delta(\sigma,x)&=-4\sigma\frac{d^2}{d\sigma^2}\Delta(\sigma,x),\end{aligned} \tag{20}$$

and some straightforward manipulations, we arrive at the result

$$\begin{aligned}\delta(x_0)[x^2h_{0i}(x)]&=\frac{-6i\partial_i\delta^4(x)}{\pi}\lim_{\sigma\to\infty}\left(\frac{\rho(\sigma)}{\sigma}\right),\\ \delta(x_0)[\vec{x}^2h_{0i}(x)]&=\frac{+5i\partial_i\delta^4(x)}{\pi}\lim_{\sigma\to\infty}\left(\frac{\rho(\sigma)}{\sigma}\right).\end{aligned} \tag{21}$$

Boulware and Jackiw[7] have emphasized that in lowest-order perturbation theory, where $\rho(\sigma)/\sigma$ does go to a constant,[8] a triple derivative of a δ function occurs in the equal-time commutator. Since the equal-time commutator is not well defined, Eqs. (21) are a more precise statement of this behavior.

The behavior $\rho(\sigma)/\sigma$ going to a constant is the case in any exactly scale-invariant theory where the currents carry the usual dimensions, as can be seen by the following argument. Let $u(s)$ be the unitary transformation causing a dilatation by the factor s,

$$u(s)j_\mu(x)u(s)^{-1}=s^d j_\mu(sx)\,. \qquad (22)$$

Here d is the dimension of the current. We require $u(s)|0\rangle=|0\rangle$. Using this operator in Eq. (2) gives

$$W_{\mu\nu}(q)=s^{2d-4}W_{\mu\nu}(q/s)\,. \qquad (23)$$

This implies

$$\rho(q^2)=s^{2d-4}\rho(q^2/s^2) \qquad (24)$$

or

$$\rho(q^2)=A(q^2)^{d-2}\,. \qquad (25)$$

Note that using this argument on different components of $W_{\mu\nu}(q)$ requires d to be the same for all Lorentz components of the current. If we have the naive dimension $d=3$, as would be required by usual current algebra, then $\rho(\sigma)/\sigma$ goes to a constant as σ goes to infinity. This means that C of Eq. (10) is undefined and a term like that in Eq. (21) is required. Recently Wolsky[9] claims to have shown that naive dimensionality for all components of the current is impossible. However, to get this result he implicitly assumes that C is finite.

III. THE SINGLE-PARTICLE EXPECTATION VALUE

We now consider the diagonal one-particle matrix element of the commutator of the electromagnetic current with itself. Define $h_{\mu\nu}(x,p)$ by

$$h_{\mu\nu}(x,p)=\langle p|[j_\mu(x),j_\nu(0)]|p\rangle\,. \qquad (26)$$

The particle $|p\rangle$ has mass m, $p^2=m^2$. We use a covariant normalization of the one-particle state with $\langle p'|p\rangle=(2\pi)^3 2p_0\delta^3(p'-p)$. In what follows if $|p\rangle$ has spin, we consider all equations as averaged over the spin states of $|p\rangle$. We shall also consider Eq. (26) as having the vacuum expectation value subtracted; in other words, $[j_\mu(x),j_\nu(0)]$ really means $[j_\mu(x),j_\nu(0)]-\langle 0|[j_\mu(x),j_\nu(0)]|0\rangle$.

Symmetry principles place several constraints on $h_{\mu\nu}(x,p)$. Translation invariance tells us

$$h_{\mu\nu}(x,p)=-h_{\nu\mu}(-x,p)\,. \qquad (27)$$

Hermiticity of $j_\mu(x)$ gives

$$h_{\mu\nu}(x,p)=-h^*_{\mu\nu}(x,p)\,. \qquad (28)$$

Time reversal times parity (TP) gives

$$h_{\mu\nu}(x,p)=+h^*_{\mu\nu}(-x,p)\,. \qquad (29)$$

Parity alone gives

$$h_{\mu\nu}(x,p)=(-1)^{g_{\mu 0}+g_{\nu 0}}h_{\mu\nu}(x_0,-\vec{x};p_0,-\vec{p})\,. \qquad (30)$$

Charge conjugation relates $h_{\mu\nu}(x,p)$ to a different process unless $|p\rangle$ is self-conjugate, in which case charge conjugation is automatically conserved here since we have two electromagnetic currents.

Given the above symmetries, we can write the most general form allowed for the equal-time commutator. Since these symmetries imply $h_{\mu\nu}(x,p)=-h_{\mu\nu}(-x,p)$, the equal-time commutator can only contain terms with an odd number of derivatives on a spatial δ function. Arbitrarily assuming only first derivatives, we can write (as in Sec. II, n_μ is any timelike vector with $n_0>0$)

$$\sqrt{n^2}\,\delta(n\cdot x)h_{\mu\nu}(x,p)=iC_{\mu\nu\alpha}(n/\sqrt{n^2},p)\partial_\alpha\delta^4(x)\,. \qquad (31)$$

If $h_{\mu\nu}(x)$ is to have a well-defined equal-time limit, the combinations of derivatives taken in Eq. (31) must involve no time derivatives in a frame where n has only a time component. This means we must require

$$n_\alpha C_{\mu\nu\alpha}(n/\sqrt{n^2},p)=0\,. \qquad (32)$$

The symmetry conditions require

$$C_{\mu\nu\alpha}=C_{\nu\mu\alpha}\,. \qquad (33)$$

We have not yet used current conservation. This constraint can be written $\partial_\mu h_{\mu\nu}(x)=\partial_\nu h_{\mu\nu}(x)=0$. Taking the divergence of (31) and using current conservation gives

$$n_\mu\delta'(n\cdot x)h_{\mu\nu}(x,p)=i\frac{1}{\sqrt{n^2}}C_{\mu\nu\alpha}(n/\sqrt{n^2},p)\partial_\mu\partial_\alpha\delta^4(x)\,. \qquad (34)$$

Let us also note that

$$\frac{d}{dn_\lambda}\,\delta(n\cdot x)h_{\mu\nu}(x,p)=x_\lambda\delta'(n\cdot x)h_{\mu\nu}(x,p)$$
$$=i\frac{d}{dn_\lambda}\left(\frac{1}{\sqrt{n^2}}C_{\mu\nu\alpha}(n/\sqrt{n^2},p)\partial_\alpha\delta^4(x)\right)\,. \qquad (35)$$

Multiplying Eq. (34) by x_λ and Eq. (35) by n_μ with a sum over the index μ gives

$$\frac{1}{\sqrt{n^2}}C_{\mu\nu\alpha}x_\lambda\partial_\mu\partial_\alpha\delta^4(x)=n_\mu\frac{d}{dn^\lambda}\frac{1}{\sqrt{n^2}}C_{\mu\nu\alpha}\partial_\alpha\delta^4(x)\,. \qquad (36)$$

This is equivalent to

$$C_{\mu\nu\alpha}=-\sqrt{n^2}\frac{d}{dn_\alpha}\left(\frac{n_\lambda}{\sqrt{n^2}}C_{\lambda\nu\mu}\right)\,. \qquad (37)$$

Now proceeding similarly using $\partial_\nu h_{\mu\nu}=0$ we get

$$C_{\mu\nu\alpha}=-\sqrt{n^2}\frac{d}{dn_\alpha}\left(\frac{n_\lambda}{\sqrt{n^2}}C_{\mu\lambda\nu}\right)\,. \qquad (38)$$

Taking the difference of these equations we find that the quantity

$$\frac{n_\lambda}{\sqrt{n^2}}C_{\lambda\nu\mu}-\frac{n_\lambda}{\sqrt{n^2}}C_{\mu\lambda\nu}$$

cannot depend on n. Because of Eq. (33) this quantity is antisymmetric under interchange of μ and ν. However no antisymmetric tensors can be constructed from p_μ alone, so this quantity must vanish. Thus we can define a symmetric tensor $C_{\mu\nu}$ by

$$\begin{aligned} C_{\mu\nu} &= -\frac{n_\lambda}{\sqrt{n^2}} C_{\mu\lambda\nu} \\ &= -\frac{n_\lambda}{\sqrt{n^2}} C_{\lambda\nu\mu} . \end{aligned} \tag{39}$$

By Eq. (32) we have

$$\begin{aligned} n_\mu C_{\mu\nu} &= n_\nu C_{\mu\nu} \\ &= 0 . \end{aligned} \tag{40}$$

From (37) or (38) we can get $C_{\mu\nu\alpha}$ from $C_{\mu\nu}$:

$$C_{\mu\nu\alpha} = +\sqrt{n^2}\frac{d}{dn_\alpha} C_{\mu\nu} . \tag{41}$$

Finally note that any $C_{\mu\nu}(n/\sqrt{n^2}, p)$ satisfying Eq. (40) will define, through Eq. (41), a $C_{\mu\nu\alpha}$ which will satisfy the constraints of current conservation. This means that the most general form for $C_{\mu\nu\alpha}$ is given by Eq. (41) with a $C_{\mu\nu}$ of the form

$$\begin{aligned} C_{\mu\nu}(n/\sqrt{n^2}, p) = &-C\left(\frac{p\cdot n}{\sqrt{n^2}}\right)\left(g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2}\right) \\ &+ D\left(\frac{p\cdot n}{\sqrt{n^2}}\right)\left(p_\mu - \frac{n_\mu p\cdot n}{n^2}\right)\left(p_\nu - \frac{n_\nu p\cdot n}{n^2}\right) . \end{aligned} \tag{42}$$

This shows the remarkable result that the constraint of current conservation determines the equal-time commutator in terms of only two invariant functions. Combining Eqs. (31) and (41) with Eq. (42) gives the most general form for the equal-time commutator consistent with our assumption of only first derivatives of spatial δ functions:

$$\sqrt{n^2}\,\delta(n\cdot x) h_{\mu\nu}(x, p) = i\sqrt{n^2}\left(\frac{d}{dn}\cdot\frac{d}{dx}\right) C_{\mu\nu}(n/\sqrt{n^2}, p)\delta^4(x) . \tag{43}$$

In Appendix A we carry out this differentiation to find the equal-time commutator explicitly in terms of C and D.

With this form for the equal-time commutator, we can find our desired restrictions on the time derivative of the commutator. Differentiating (43) with respect to n_λ gives

$$x_\lambda \delta'(n\cdot x) h_{\mu\nu}(x, p) = i\frac{d}{dn_\lambda}\left(\frac{d}{dn}\cdot\frac{d}{dx}\right) C_{\mu\nu}(n/\sqrt{n^2}, p)\delta^4(x) . \tag{44}$$

This can be true only if

$$\begin{aligned} \delta'(n\cdot x) h_{\mu\nu}(x, p) = &-\frac{i}{2}\left(\frac{d}{dn}\cdot\frac{d}{dx}\right)^2 C_{\mu\nu}(n/\sqrt{n^2}, p)\delta^4(x) \\ &-\frac{1}{n^2} i K_{\mu\nu}(n/\sqrt{n^2}, p)\delta^4(x) . \end{aligned} \tag{45}$$

Here $K_{\mu\nu}$ is an undetermined dynamical tensor analogous to that found in Sec. II. As in the vacuum case, current conservation requires

$$\begin{aligned} n_\mu K_{\mu\nu} &= n_\nu K_{\mu\nu} \\ &= 0 . \end{aligned} \tag{46}$$

This means $K_{\mu\nu}$ must have the form

$$\begin{aligned} K_{\mu\nu} = &A\left(\frac{p\cdot n}{\sqrt{n^2}}\right)\left(g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2}\right) \\ &+ B\left(\frac{p\cdot n}{\sqrt{n^2}}\right)\left(p_\mu - \frac{n_\mu p\cdot n}{n^2}\right)\left(p_\nu - \frac{n_\nu p\cdot n}{n^2}\right) . \end{aligned} \tag{47}$$

It is possible that $K_{\mu\nu}$ is a divergent quantity. If this is the case, the time derivative of $h_{\mu\nu}(x)$ is not well defined at $x_0 = 0$. As in Sec. II, we can now write the time derivative of the commutator:

$$\begin{aligned} \delta(n\cdot x)\partial\cdot n h_{\mu\nu}(x, p) &= -n^2\delta'(n\cdot x) h_{\mu\nu}(x, p) + \partial\cdot n\delta(n\cdot x) h_{\mu\nu}(x, p) \\ &= i\left[\left(n\cdot\frac{d}{dx}\right)\left(\frac{d}{dn}\cdot\frac{d}{dx}\right) + \tfrac{1}{2}n^2\left(\frac{d}{dn}\cdot\frac{d}{dx}\right)^2\right] C_{\mu\nu}(n/\sqrt{n^2}, p)\delta^4(x) + iK_{\mu\nu}\,\delta^4(x) . \end{aligned} \tag{48}$$

In Appendix B we carry out the above differentiations to give the derivative of the commutator explicitly in terms of A, B, C, and D. Things simplify considerably if we go to the rest frame of $|p\rangle$ and let $n_\mu = g_{\mu 0}$. Table I summarizes our results on the commutator and its time derivative in this frame.

Now that we have expressions for the commutator and its time derivative at equal times, let us try to relate them to the total cross sections for hadron production by virtual photons incident on $|p\rangle$.

Introduce the tensor $W_{\mu\nu}(q, p)$ and the invariant functions W_1 and W_2 by

$$\begin{aligned} W_{\mu\nu}(q, p) &= \int d^4x\, e^{iq\cdot x} h_{\mu\nu}(x, p) \\ &= \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(\nu, q^2) \\ &\quad + \left(p_\mu - \frac{q_\mu p\cdot q}{q^2}\right)\left(p_\nu - \frac{q_\nu p\cdot q}{q^2}\right) W_2(\nu, q^2) . \end{aligned} \tag{49}$$

Here ν is the photon energy in the lab frame, given by

$$\nu = \frac{p \cdot q}{m}. \tag{50}$$

We have the crossing property

$$W_{1,2}(\nu, q^2) = -W_{1,2}(-\nu, q^2). \tag{51}$$

Define σ_L (σ_T) to be the total cross section up to a factor of e^2 for hadronic production by longitudinally (transversely) polarized virtual photons on $|p\rangle$. Then we have

$$\begin{aligned} \sigma_T(\nu, q^2) &= -\frac{1}{4mq_3} W_1(\nu, q^2), \\ \sigma_L(\nu, q^2) &= \frac{1}{4mq_3 |q^2|}(-q^2 W_1 + m^2 q_3^2 W_2). \end{aligned} \tag{52}$$

Let us work in a Lorentz frame where $p = (m, 0, 0, 0)$ and $q = (q_0, 0, 0, q_3)$. We relate the equal-time commutator in this frame to the above cross sections through

$$\int_{-\infty}^{\infty} dq_0 W_{\mu\nu}(q, p)|_{q_3} = 2\pi \int d^4x\, e^{-iq_3 x_3} \delta(x_0) h_{\mu\nu}(x, p). \tag{53}$$

Instead of ν and q^2, let us use q^2 and q_3 as our independent variables. Using Table I we get the result

$$\int_{-q_3^2}^{\infty} dq^2 \frac{\sigma_L(q^2, q_3)}{|q^2|} = \frac{\pi C}{2q_3 m}. \tag{54}$$

Here C means $C(p_0)|_{p_0=m}$. This result has been obtained previously by others.[10,11]

We now relate the cross sections to the time derivatives of the commutator using

$$\int_{-\infty}^{\infty} dq_0 q_0 W_{\mu\nu}(q, p) = 2\pi i \int d^4x\, e^{-iq_3 x_3} \delta(x_0) \partial_0 h_{\mu\nu}(x, p). \tag{55}$$

TABLE I. Values for the commutator and its time derivative at equal times in the external-particle rest frame. All invariant functions are evaluated at $p \cdot n / \sqrt{n^2} = m$. C' denotes $(d/dp_0) C(p_0)|_{p_0=m}$.

$$\begin{aligned}
&\delta(x) h_{00}(x) = 0 \\
&\delta(x_0) h_{0i}(x) = iC\partial_i \delta^4(x) \\
&\delta(x_0) h_{ij}(x) = 0 \\
&\delta(x_0)\partial_0 h_{00}(x) = -iC(\Box - \partial_0^2)\delta^4(x) = +iC\vec{\nabla}^2\delta^4(x) \\
&\delta(x_0)\partial_0 h_{0i}(x) = 0 \\
&\delta(x_0)\partial_0 h_{ij}(x) = i[(C + m^2 D)\partial_i\partial_j + C' m \tfrac{1}{2}(\Box - \partial_0^2) g_{ij} + A g_{ij}]\delta^4(x) \\
&\qquad = i[(C + m^2 D)\partial_i\partial_j - \tfrac{1}{2}C' m \vec{\nabla}^2 g_{ij} + A g_{ij}]\delta^4(x)
\end{aligned}$$

Looking at various components of this equation gives three independent nontrivial relations, one of which is equivalent to Eq. (54). The other two relations, which are our new results, are

$$\int_{-q_3^2}^{\infty} dq^2 \sigma_L(q^2, q_3) \frac{|q^2|}{q^2}\Big|_{q_3} = \frac{\pi}{2m}\left((m^2 D + \tfrac{1}{2} m C') q_3 + \frac{A}{q_3}\right), \tag{56}$$

$$\int_{-q_3^2}^{\infty} dq^2 \sigma_T(q^2, q_3)|_{q_3} = \tfrac{1}{4}\pi C' q_3 + \frac{\pi}{2m q_3} A. \tag{57}$$

If the particle $|p\rangle$ is assumed to be stable to electromagnetic decay, the lower limit of integration in Eqs. (54), (56), and (57) can be raised to $2m[m - (m^2 + q_3^2)^{1/2}]$.

Let us now discuss how these results relate to possible q-number Schwinger terms. Such terms manifest themselves in our formalism through the nonvanishing of $C(p_0)$ or $D(p_0)$ for some value of p_0. We have removed any non-q-number Schwinger terms by considering the commutator with the vacuum expectation value subtracted off. Of course if $C'(m)$ or $D'(m)$ is nonzero, $C(p_0)$ or $D(p_0)$ cannot vanish identically. Present experimental data indicate that $\sigma_T(q^2, q_3)$ goes to a nonzero constant value as q_3 becomes large with q^2 fixed.[12] The value of this constant is dependent on q^2. Letting q_3 become large in Eq. (57) would then seem to indicate that C' is nonvanishing and there are q-number Schwinger terms. Assuming the constant asymptotic cross-section behavior is correct, there are three ways to avoid this conclusion.

One way is to have $\int_{-\infty}^{\infty} dq^2 \sigma_T(q^2, q_3 = \infty) = 0$. If this were the case, the asymptotic cross section would have to be negative for some values of q^2. This does not violate any positivity condition since we have subtracted the vacuum expectation value of the commutator, which is an infinite positive quantity for timelike q^2. Indeed vector-meson dominance suggests that $\sigma_T(q^2 = m_\rho^2, q_3 = \infty)$ should be a negative quantity.

A second way of avoiding the conclusion that $C'(m)$ does not vanish is to have important negative contributions to the integral in Eq. (57) coming from $q^2 \gtrsim q_3 m$ even as q_3 becomes large. Here lowest-order perturbation theory with fermions gives a negative cross section; however, this negative contribution to the integral in Eq. (57) is canceled by contributions from $q^2 \approx -q_3 m$ as q_3 becomes large. Similarly, if in this kinematic region the state $|p\rangle$ can be considered as a bound state of point particles, as has been argued by several authors,[13] this contribution again will be unimportant as q_3 becomes large. We will see this cancellation in more detail later when we discuss the Bjorken scaling region. Thus we feel it unlikely that large negative contributions to the integral in Eq. (55) can come from this region of $q^2 \gtrsim q_3 m$ for large q_3.

The third way of having $C' = 0$ is to have $A(m)$ be a divergent quantity. This requires the integral in Eq. (57) to diverge for all values of q_3. This means that, at any fixed q_3, $\sigma_T(q^2, q_3)$ cannot fall faster than $1/q^2$ as q^2 becomes large compared to $q_3 m$. Although we cannot disprove such behavior, it does not occur in lowest-order perturbation theory and we regard such behavior as unlikely.

On the basis of this discussion we conclude that it would be simplest to either have q-number Schwinger terms with $C'(m)$ nonzero or to have the sum rule

$$\int_{-\infty}^{\infty} dq^2 \sigma_T(q^2, q_3 = \infty) = 0 .$$

If the parameter $C'(m)$ does not vanish, then the above integral must diverge. Of course, the vanishing of this integral does not necessarily imply an absence of operator Schwinger terms; it would only imply $C'(m) = 0$.

Bjorken[14] has conjectured that, in the limit of large q^2 and q_3 with the ratio $q^2/q_3 \equiv \omega$ constant, σ_T and σ_L have the behavior

$$\sigma_{T,L} = -\frac{1}{q^2} f_{T,L}(\omega) . \tag{58}$$

Let us now discuss the contributions to our integral relations from this Bjorken scaling region. For spacelike q^2 the vacuum subtraction does not alter the fact that the total cross sections are squares, so we have

$$f_{T,L}(\omega) > 0 \quad \text{for} \quad \omega < 0 . \tag{59}$$

From the implicit assumption of a well-defined commutator, Stack[10] has shown that

$$\begin{aligned} f_T(\omega) &= +f_T(-\omega) , \\ f_L(\omega) &= -f_L(-\omega) . \end{aligned} \tag{60}$$

Because of the crossing property (60) this Bjorken region gives vanishing contributions to Eqs. (56) and (57). To use this scaling in Eq. (54), let us assume vanishing of $\sigma_L(q^2, q_3)$ as $q_3 \to \infty$ with q^2 fixed. Experimental data are not yet sufficiently accurate to comment on this assumption. We can then assume that Eq. (54) is dominated by the Bjorken scaling region. This gives the result

$$\int_{-2m}^{0} \frac{f_L(\omega) d\omega}{\omega^2} = +\frac{\pi C}{4m} . \tag{61}$$

Since $f_L(\omega)$ is non-negative for ω in the integration region, we must have $C \neq 0$ unless $f_L(\omega)$ vanishes identically. This result has been obtained previously by others.[10,11] Our assumption of $\sigma_L(q^2, \infty) = 0$ and dominance by the Bjorken scaling region can be inserted in Eq. (56) with the result

$$m^2 D + \tfrac{1}{2} m C' = 0 . \tag{62}$$

We stress that $\sigma_L(q^2, q_3)$ could vanish even in the presence of q-number Schwinger terms; $C'(m)$ could be nonvanishing even if $C(m)$ and $m^2 D + \frac{1}{2} m C'$ vanish. Finally we note that because of Eq. (60) the Bjorken scaling region does not contribute to Eq. (57) in the limit of large q_3 so this region will not affect our discussion of $C'(m)$.

IV. CONCLUSION

Using a technique of Furlan and Rossetti, and Cornwall and Norton,[3] we have studied the commutator of the time derivative of the electromagnetic current with the electromagnetic current at equal times. We find that the vacuum expectation value of this quantity must contain second derivatives of spatial δ functions. These terms are completely determined by the vacuum expectation value of the equal-time commutator of two electromagnetic currents. Beyond Lorentz invariance and current conservation our only assumption is that this equal-time commutator is well defined.

We then discussed the one-particle matrix element of the same commutators. We obtained relations between possible q-number Schwinger terms and the total cross sections for hadron production with virtual photons incident on the one-particle state. Representing the transverse cross section for photon mass squared q^2 and lab-frame (rest frame of p) momentum $|\vec{q}|$ by $\sigma_T(q^2, |\vec{q}|)$, we find that at least one of the following is true:

(1) There are q-number Schwinger terms.

(2) $\int_{-\infty}^{\infty} dq^2 \sigma_T(q^2, |\vec{q}| = \infty) = 0$.

(3) The integral $\int_{-q_3^2}^{\infty} dq^2 \sigma_T(q^2, |\vec{q}|)$ for increasing $|\vec{q}|$ has increasing contributions from q^2 of the order or larger than $|\vec{q}|$. We include here the possibility that this integral diverges for all $|\vec{q}|$.

In the above we work with the vacuum expectation value subtracted from the commutators.

We must finally emphasize that all of our relations on the virtual photoabsorption cross section involve to some extent timelike photons. These cross sections are experimentally unmeasurable. Thus our results are of no direct experimental interest. To get relations on experimental quantities from equal-time commutators either one must take an infinite-momentum limit, which actually corresponds to evaluating the commutator on the light cone,[15] or one must make subtraction assumptions on dispersion relations. We have not done this here. Our results can, however, be of use in checking the consistency of theoretical models which do predict the timelike cross sections.

APPENDIX A

Here we give the most general form for the one-particle diagonal matrix element of the commutator of two currents. It is obtained by carrying out the differentiation indicated in Eq. (43) using $C_{\mu\nu}$ as given in Eq. (42) of the same section. This gives the result

$$
\begin{aligned}
\sqrt{n^2}\,\delta(n\cdot x)\langle p|[j_\mu(x),j_\nu(0)]|p\rangle
&= \sqrt{n^2}\,\delta(n\cdot x)h_{\mu\nu}(x) \\
&= i\Bigg[C\left(\frac{p\cdot n}{\sqrt{n^2}}\right)\left(\frac{n_\mu\partial_\nu+n_\nu\partial_\mu}{\sqrt{n^2}}-\frac{2n_\mu n_\nu\partial\cdot n}{(\sqrt{n^2})^3}\right)+D\left(\frac{p\cdot n}{\sqrt{n^2}}\right)\Bigg(-\frac{(p_\mu\partial_\nu+p_\nu\partial_\mu)p\cdot n}{\sqrt{n^2}}-\frac{(p_\mu n_\nu+p_\nu n_\mu)p\cdot\partial}{\sqrt{n^2}} \\
&\qquad +\frac{2(p_\mu n_\nu+p_\nu n_\mu)p\cdot n\,n\cdot\partial}{(\sqrt{n^2})^3}+\frac{(n_\mu\partial_\nu+n_\nu\partial_\mu)(p\cdot n)^2}{(\sqrt{n^2})^3}+\frac{2n_\mu n_\nu p\cdot n\,p\cdot\partial}{(\sqrt{n^2})^3}-\frac{4n_\mu n_\nu(p\cdot n)^2 n\cdot\partial}{(\sqrt{n^2})^5}\Bigg) \\
&\qquad +C'\left(\frac{p\cdot n}{\sqrt{n^2}}\right)\left(-g_{\mu\nu}p\cdot\partial+g_{\mu\nu}\frac{p\cdot n\,\partial\cdot n}{n^2}+\frac{n_\mu n_\nu p\cdot\partial}{n^2}-\frac{n_\mu n_\nu p\cdot n\,\partial\cdot n}{(n^2)^2}\right) \\
&\qquad +D'\left(\frac{p\cdot n}{\sqrt{n^2}}\right)\Bigg(p_\mu p_\nu\partial\cdot p-\frac{p_\mu p_\nu p\cdot n\,\partial\cdot n}{n^2}-\frac{(p_\mu n_\nu+p_\nu n_\mu)p\cdot n\,\partial\cdot p}{n^2} \\
&\qquad\qquad +\frac{(p_\mu n_\nu+p_\nu n_\mu)(p\cdot n)^2\,\partial\cdot n}{(n^2)^2}+\frac{n_\mu n_\nu(p\cdot n)^2\,\partial\cdot p}{(n^2)^2}-\frac{n_\mu n_\nu(p\cdot n)^3\,\partial\cdot n}{(n^2)^3}\Bigg)\Bigg]\delta^4(x).
\end{aligned}
\tag{A1}
$$

To make this more manageable, look at various components with $n_\mu = g_{\mu 0}$:

$$\delta(x_0)h_{00}(x)=0\,, \tag{A2}$$

$$\delta(x_0)h_{0i}(x)=i[C(p_0)\partial_i - D(p_0)p_i(\partial\cdot p-\partial_0 p_0)]\delta^4(x)\,, \tag{A3}$$

$$\delta(x_0)h_{ij}(x)=i[-D(p_0)p_0(\partial_i p_j+\partial_j p_i)-C'(p_0)g_{ij}(p\cdot\partial-p_0\partial_0)+D'(p_0)p_i p_j(\partial\cdot p-\partial_0 p_0)]\delta^4(x)\,. \tag{A4}$$

Letting $p_\mu = m g_{\mu 0}$ gives the results in the first half of Table I.

APPENDIX B

Here we give the result of carrying out the differentiation indicated in Eq. (48) to obtain the time derivative of the commutator. Setting $n_\mu = g_{\mu 0}$, we get

$$\delta(x_0)\partial_0 h_{00}(x,p)=i[C(p_0)(-\Box+\partial_0{}^2)+D(p_0)(p\cdot\partial-\partial_0 p_0)^2]\delta^4(x)\,, \tag{B1}$$

$$
\begin{aligned}
\delta(x_0)\partial_0 h_{0i}(x,p)&=i\{D(p_0)[p_i p_0(\Box-\partial_0{}^2)+p_0\partial_i(p\cdot\partial-p_0\partial_0)] \\
&\qquad +C'(p_0)[\partial_i(p\cdot\partial-\partial_0 p_0)]+D'(p_0)[-p_i(p\cdot\partial-\partial_0 p_0)^2]\}\,\delta^4(x)\,,
\end{aligned}
\tag{B2}
$$

$$
\begin{aligned}
\delta(x_0)\partial_0 h_{ij}(x,p)&=i\{C(p_0)(\partial_i\partial_j)+D(p_0)[p_0{}^2\partial_i\partial_j-(p_i\partial_j+p_j\partial_i)(p\cdot\partial-\partial_0 p_0)] \\
&\qquad +C'(p_0)[\tfrac{1}{2}g_{ij}p_0(\Box-\partial_0{}^2)]+D'(p_0)[-p_0(p_i\partial_j+p_j\partial_i)(p\cdot\partial-p_0\partial_0)-\tfrac{1}{2}p_i p_j p_0(\Box-\partial_0{}^2)] \\
&\qquad +C''(p_0)[-\tfrac{1}{2}g_{ij}(p\cdot\partial-\partial_0 p_0)^2]+D''(p_0)[-\tfrac{1}{2}p_i p_j(p\cdot\partial-\partial_0 p_0)^2]+A(p_0)g_{ij}+B(p_0)p_i p_j\}\,\delta^4(x)\,.
\end{aligned}
\tag{B3}
$$

If we set $p_\mu = m g_{\mu 0}$ we get the second half of Table I.

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Loop Graph in the Dual-Tube Model*

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The one-loop graph in the dual-tube model is constructed. The conditions for no divergences or new singularities are exactly those found by Lovelace for factorization of the "Pomeranchukon" in the strip model. Loops correspond to electrostatics on multiple tori only if spurious particles are permitted to circulate in the loops.

The usual dual-resonance model can be understood in terms of an electrostatic analog in which the ether is a two-dimensional strip.[1] Resonances correspond to long, thin strips and loop diagrams to annuli. External particles correspond to charges on the edges of the strip. Singularities in the scattering amplitude are associated either with an accumulation of charges corresponding to external particles, or to singularities in the shape of the ether surface.[2] For example, the one-loop diagram corresponds to an annulus of ether with either all charges on one boundary (planar loop) or some on each (nonplanar). When the hole shrinks to zero, one gets a divergence[3] (planar case) or the "Pomeranchukon" singularity[4] (nonplanar).

Almost a year ago I proposed a model[5] with a different kind of duality than that of the strip model. In the strip model only planar channels are dual to each other, while in the new model all channels are dual. The ether is a closed two-dimensional surface (a sphere for the tree diagrams) and resonances correspond to tubes instead of strips, with external particles entering as charges anywhere on the surface. I conjectured that higher-order diagrams would correspond to electrostatics on multiple tori (spheres with n handles). It was further conjectured that the difficulties in the strip model coming from shrinking the hole in the annulus might disappear in this model.

In this article I calculate the one-loop diagram in the tube model. We shall see that the electrostatic analog applies only if one permits spurious states to circulate in the loop, but that, whether or not they are permitted, there exist choices of dimensions and assumed Virasoro-type gauges for which there are no divergences or new singularities. For the case where these spurious states are projected out, the dimensionalities are exactly those Lovelace found for the factorization