

Lattice gauge notes IX

The Renormalization Group

Peter keeps asking how does the Lorentz invariance, or rotational invariance in Euclidian space, get restored in the lattice. This is not entirely automatic, but requires a continuum limit. However for renormalizable theories this continuum limit is quite naturally invariant under the desired continuum symmetries. This is dimension dependent and tied to the fact that there are only a finite number of parameters that need to be tuned.

Before I get more “sophisticated,” I give the simple argument for free fields. On the lattice powers of the momentum get replaced by trigonometric functions. Thus for a scalar field I have

$$\begin{aligned} D(p) &= \frac{a^2}{\sum_{\mu}(1 - \cos(ap_{\mu})) + a^2 m^2} \\ &= \frac{1}{p^2 + m^2} + \frac{a^2(p^2 + m^2) - (\sum_{\mu}(1 - \cos(ap_{\mu})) + a^2 m^2)}{(p^2 + m^2)(\sum_{\mu}(1 - \cos(ap_{\mu})) + a^2 m^2)} \\ &= \frac{1}{p^2 + m^2} - \frac{a^2 \sum_{\mu} p_{\mu}^4}{(p^2 + m^2)^2} + O(a^4) \end{aligned}$$

Thus for small momentum, the propagator gets a non-Lorentz invariant piece proportional to $\sum_{\mu} p_{\mu}^4$, but it is of order a^2 . Note that the natural lattice mass is am and for the continuum limit this is “tuned” to zero to give the desired physical mass.

(adapted from my rhic97 notes):

I now turn to the renormalization group, which I approach via the Migdal-Kadanoff approximate recursion relations. I start with a discussion of decimation. I begin with the Ising model in a magnetic field, with partition function

$$Z = \sum_{s_i} e^{\beta \sum_{nn} s_i s_j + H \sum_i s_i}$$

Consider this first in one dimension, i.e. consider a chain of sites in a ring, ultimately letting the ring length go to infinity. A bond in the chain connecting s and s' contributes

$$T_{s,s'} = e^{\beta s s' + H(s+s')/2} = \begin{pmatrix} e^{\beta+H} & e^{-\beta} \\ e^{-\beta} & e^{\beta-H} \end{pmatrix}_{s,s'}$$

This is the “transfer matrix.” Summing over spins gives the partition function for an N site lattice

$$Z = \text{Tr}(T^N)$$

This can be calculated by diagonalizing T

$$Z = \lambda_+^N + \lambda_-^N$$

with

$$\lambda_{\pm} = e^{\beta} \left(\cosh(H) \pm \sqrt{\sinh^2(H) + e^{-4\beta}} \right)$$

The free energy per site is dominated by the largest eigenvalue

$$-\beta F = \frac{\log Z}{N} = \log(\lambda_+) + \exp(-N \log(\lambda_+/\lambda_-)) + \dots$$

Since λ_+ is analytic and positive, the infinite volume free energy has no singularities. We fail to find a phase transition. Note that the finite volume corrections are exponentially small. This shows that the theory has a mass gap. As β goes to infinity with $H = 0$, the eigenvalues become equal and the mass gap goes to zero. The phase transition, to the extent there is one, occurs at zero temperature.

Physically, a kink anti-kink pair has a finite probability, but an infinite number of possible separations. This infinity always disorders the system. In more dimensions a big bubble pays a penalty proportional to its surface, so it is suppressed.

Now I solve this another way. Let me sum over every other spin, giving

$$Z = \text{Tr}(T')^{N/2}$$

where

$$T' = T^2 = \begin{pmatrix} e^{2(\beta+H)} + e^{-2\beta} & e^H + e^{-H} \\ e^H + e^{-H} & e^{2(\beta-H)} + e^{-2\beta} \end{pmatrix}$$

We now match this with the original form of T

$$T' = C \begin{pmatrix} e^{\beta'+H'} & e^{-\beta'} \\ e^{-\beta'} & e^{\beta'-H'} \end{pmatrix}$$

We see that exactly the same physics occurs on a lattice of twice the spacing and new couplings (β', H') . The values of C , β' and H' are fixed by the three equations

$$\begin{aligned} C e^{-\beta'} &= e^H + e^{-H} \\ C e^{\beta'+H'} &= e^{2(\beta+H)} + e^{-2\beta} \\ C e^{\beta'-H'} &= e^{2(\beta-H)} + e^{-2\beta} \end{aligned}$$

This process is called decimation, *i.e.* integrating out some of the degrees of freedom. To simplify the equations, I turn off H , obtaining

$$\beta' = \frac{1}{2} \log(\cosh(2\beta))$$

This can be written in a form

$$\beta' - \beta = -\frac{1}{2} \log \left(\frac{2}{1 + e^{-4\beta}} \right)$$

A fixed point would have the new and old couplings equal, the only such here occurs at $\beta = 0$. The new coupling is always less than the old one as long as beta is positive. Repeating this as an iteration drives any β to zero.

It is instructive to extend this to non integer decimations. For this, write the transfer matrix in the form $T_{ss'} = \cosh(\beta)(1 + ss' \tanh(\beta))$. The above decimation by a factor of two involves the sum

$$\frac{1}{2} \sum_{s_2} (1 + s_1 s_2 t)(1 + s_2 s_3 t) = (1 + s_1 s_3 t^2)$$

or simply $\tanh(\beta) \rightarrow \tanh^2(\beta)$. Interpolate this to rescaling by a factor of $1 + \Delta$, taking $\tanh(\beta) \rightarrow \tanh^{1+\Delta}(\beta)$. Infinitesimally, this reduces to

$$\frac{\beta' - \beta}{\Delta} \sim a \frac{d\beta}{da} = \cosh(\beta) \sinh(\beta) \log(\tanh(\beta))$$

This is the renormalization group equation for this system. The right hand side is negative for all positive β . As the lattice spacing varies from zero to infinity, the coupling β flows from the ultraviolet fixed point at infinity to the infrared fixed point at zero.

Suppose a system has a non-trivial fixed point satisfying

$$a \frac{d\beta}{da} = \lambda(\beta - \beta_c).$$

This has the solution

$$\beta = \beta_c + C a^\lambda$$

Since $a \sim 1/\xi$, this says

$$1/\xi \sim (\beta - \beta_c)^{1/\lambda}$$

This is the renormalization group way of seeing how non-trivial exponents can arise as one approaches a critical point.

Going on to more dimensions we lose the exactness and must make approximations. Integrating out a site in more than one dimension introduces couplings between all sites to which it is coupled. Integrating the sites along a line couples all spins attached to that line. Integrating out all but the corners on a block requires an infinite number of couplings. This makes things less than rigorous, but can imagine a similar coupling “flow” in a higher space.

Making an approximation by moving bonds around allows one to analytically study these flows. Imagine integrating out every other site in say the x direction. To avoid long range couplings being generated in the y direction, follow Kadanoff and “move” the y bonds to sites not being integrated over. Every second y bond becomes twice as strong, and then

the earlier x decimation can be carried out on the remaining sites. Thus we relate the model at β_x, β_y to that at

$$\begin{aligned}\beta'_x &= \frac{1}{2} \log(\cosh(2\beta_x)) \\ \beta'_y &= 2\beta_y\end{aligned}$$

Now repeat this for the y direction. The resulting transformation is asymmetric due to the approximations. To get more symmetric, do things differentially, using the earlier equation for the x coupling and $a \frac{d\beta}{da} = \beta$ for the bond moving. The total change of coupling is then

$$a \frac{d\beta}{da} = \cosh(\beta) \sinh(\beta) \log(\tanh(\beta)) + (d-1)\beta$$

Here I insert a factor of $d-1$ to allow for bond moving in all directions but the decimation one. The result is exact in one dimension, and for $d=2$ it still gives the exact β_c .

The renormalization group relates theories with different lattice spacings. If we could keep track of an infinite number of couplings, the procedure would be “exact,” but in reality we usually need some truncation. Continuing to integrate out degrees of freedom, the couplings flow and might reach some “fixed point.” With two couplings, there can be an attractive “sheet” towards which couplings flow, and then they go towards the fixed point. If the fixed point has only one attractive direction, then two different models that flow towards that same fixed point will have the same physics. This is universality, *i.e.* exponents are the same for all these models with the same attractor.

Some remarkable conclusions come from dimensional analysis, although, in ignoring non-perturbative effects that might occur at strong coupling, the following arguments are not rigorous. In d dimensions a conventional scalar field has dimensions of $M^{\frac{d-2}{2}}$. Thus the coupling constant λ in an interaction of form $\int d^d x \lambda \phi^n$ has dimensions of $M^{d-n\frac{d-2}{2}}$. On a lattice of spacing a , the natural unit of dimension M is the inverse lattice spacing. Thus without any special tuning, the renormalized coupling at some fixed physical scale would naturally run as $\lambda \sim a^{n\frac{d-2}{2}-d}$. As long as the exponent in this expression is positive, *i.e.*

$$n \geq \frac{2d}{d-2}$$

we expect the coupling to become “irrelevant” in the continuum limit. The fixed point is driven towards zero in the corresponding direction. If d exceeds four, this is the case for all interactions. (I ignore ϕ^3 in 6 dimensions because of stability problems.) This suggests that four dimensions is a critical case, with mean field theory giving the right qualitative critical behavior for all larger dimensions. In four dimensions we have several possible “renormalizable” couplings which are dimensionless, suggesting logarithmic corrections to the simple dimensional arguments. Indeed, four-dimensional non-abelian gauge theories should display exactly such a logarithmic flow; this is asymptotic freedom.

This simple dimensional argument applied to the mass term suggests it would flow towards infinity in all dimensions. For a conventional phase transition, something must be tuned to

a critical point. In statistical mechanics this is the temperature. In field theory language we usually remap this onto a tuning of the mass term, saying that the transition occurs as some scalar mass goes through zero. This tuning of scalar mass terms required for a continuum limit seems unnatural and is one of the unsatisfying features of the standard model, driving particle physicists to try to unravel how the Higg's mechanism really works.

In non-Abelian gauge theories with massless fermions, chiral symmetry protects the mass from renormalization, avoiding any special tuning. Indeed, these models exhibit the amazing phenomenon of dimensional transmutation: all dimensionless parameters in the continuum limit are completely determined by the basic structure of the initial Lagrangian, without any continuous parameters to tune. In the limit of vanishing pion mass, the rho to nucleon mass ratio should be determined from first principles; it is the goal of lattice gauge theory to calculate just such numbers.

As we go below four dimensions, this dimensional argument suggests that several couplings can become "relevant," requiring the renormalization group picture of flow towards a non-trivial fixed point. Above two dimensions the finite number of renormalizable couplings corresponds to the renormalization group argument for a finite number of "universality classes," corresponding to different basic symmetries.

Let me close with a short list of what I regard as the main unsolved problems in lattice gauge theory:

The two top ones:

- chemical potential: then $|D|$ is not positive, cannot use Monte Carlo
- chiral gauge theories: couple gauge fields to non-anomalous chiral symmetries

Then my favorites:

- fermion algorithms: remain awkward, only exact for even N_f
- supersymmetry: not natural, like chiral
- Higgs mechanism: what is the simplest way to avoid triviality

Other popular issues:

- gravity
- improved actions: coarser lattices
- get some numbers! spectra, matrix elements, ...