Lattice gauge notes VIII

Hamiltonian formulation for pure gauge

(this section from chapter 15 of my book)

Path integrals are formally justified as in the first lecture by relating then to a Hamiltonian. The path integral with periodic time is formally a four dimensional classical partition function, but it is also a quantum mechanical partition function in three dimensions, i.e. $Z = \mathrm{Tr} e^{-H/kT}$ with temperature T and Boltzmann's constant k. The variables in lattice gauge theory are somewhat more complicated than in the simple quantum mechanical model I did before, so lets try to reproduce the same steps with lattice gauge theory.

Here I want to explore the continuum time limit of the Wilson pure gauge theory. This suggests treating the spacelike and timelike lattice spacing separately. This results in separate couplings for timelike and spacelike plaquettes; so, I consider the action

$$S = -\beta_s \sum_s \mathrm{ReTr} U_p - \beta_t \sum_t \mathrm{ReTr} U_p$$

where s refers to spacelike plaquettes and t to timelike. Introducing a_0 and a to represent the timelike and spacelike lattice spacing, the formal classical relation for $SU(n)$ is

$$\beta_s = \frac{2na_0}{g_s^2 a}$$
$$\beta_t = \frac{2na}{g_t^2 a_0}$$

Quantum corrections will enter at higher order and thus I distinguish g_s and g_t. To lowest order these are equal to the bare coupling. For the moment, just note that if the timelike lattice spacing goes to zero first, β_t goes to infinity. This drives the timelike plaquettes to unity first.

I want to find a quantum mechanical Hilbert space and a "transfer matrix" operator T such that

$$Z = \int (dU) e^{-S} = \mathrm{Tr} T^N = \mathrm{Tr} e^{-H/k\mathcal{T}}$$

with N being the number of time slices for our periodic lattice and $\mathcal{T}$ the physical temperature. If we identify $Na_0 = 1/k\mathcal{T}$ our Hamiltonian is formally $H = -Nk\mathcal{T} \log T = -\frac{1}{a_0} \log T$.

Our Hilbert space is the direct product of spaces of square integrable functions over the gauge group. A state $|\psi\rangle$ is specified by a wave function $\psi(U)$ which is a function of link

variables U_{ij} which in turn are group elements associated with each bond of a spacelike lattice. The inner product is

$$\langle\psi'|\psi\rangle = \int (dU)\psi^*(U)\psi(U)$$

Here the (dU) only refers to integration over spacelike links.

A natural non-normalizable basis $\{|U\rangle\}$ consists of states specified by group elements on each bond. These elements are constrained bo that reversed links are not independent $U_{ij} = U^\dagger_{ji}$. The overlap of these basis states is

$$\langle U'|U\rangle = \prod_{ij} \delta(U'_{ij}, U_{ij})$$

where the group theoretical delta function is defined in the natural way

$$\int dg f(g)\delta(g, g') = f(g')$$

for any function $f(g)$. Completeness reads

$$1 = \int (dU)|U\rangle\langle U|$$

and a general state takes the form

$$|\psi\rangle = \int (dU)|U\rangle\psi(U)$$

I can now write down by inspection the matrix elements of my desired transfer matrix

$$\begin{aligned}\langle U'|T|U\rangle = &\exp\left(\frac{2a}{g_t^2 a_0}\sum_{ij}\mathrm{ReTr}(U^\dagger_{ij}U'_{ij})\right)\\ &\times\exp\left(\frac{2a_0}{g_s^2 a}\sum_s \mathrm{ReTr}U_s\right)\end{aligned}$$

Just as for ordinary quantum mechanics we have conjugate operators $\hat{p}$ and $\hat{x}$, we want to find a complete set of operators to generate our Hilbert space. In analogy with $\hat{x}$, define a set of matrix valued operators $\hat{U}_{ij}$ satisfying

$$\hat{U}_{ij}|U\rangle = U_{ij}|U\rangle$$

The conjugate variable should generate the analog of translations, and indeed it convenient to start directly with the translation operators. Thus given some group element g and some link (ij), consider the quantum operator $R_{ij}(g)$ defined by

$$R_{ij}(g)|U\rangle = |U'\rangle$$

where $U'_{ij} = gU_{ij}$ and on all other links the U's are unchanged. These group-translation operators satisfy the group representation property

$$R_{ij}(g)R_{ij}(g') = R_{ij}(gg')$$

Using these operators, the transfer matrix takes the form

$$\begin{aligned}\langle U'|T|U\rangle = &\prod_{ij}\int dgR_{ij}(g)\exp\left(\frac{2a}{g_t^2 a_0}\sum_{ij}\mathrm{ReTr}g\right)\\ &\times\exp\left(\frac{2a_0}{g_s^2 a}\sum_s \mathrm{ReTr}U_s\right)\end{aligned}$$

Now for some more group theory. Since $R_{ij}(g)$ represents the group, let me write it in terms of generators. Thus consider

$$g = e^{i\omega^\alpha\lambda^\alpha} = e^{i\omega\cdot\lambda}$$

where ω^α are a set of parameters and λ^α the fundamental group generators, which satisfy

$$\begin{aligned}&\lambda^\dagger = \lambda\\ &\mathrm{Tr}\lambda^\alpha\lambda^\beta = \frac{1}{2}\delta^{\alpha\beta}\\ &[\lambda^\alpha,\lambda^\beta] = if^{\alpha\beta\gamma}\lambda^\gamma\end{aligned}$$

With some foresight in notation, I introduce operators E^α_{ij} which generate R through the equation

$$R_{ij}(g) = e^{i\omega^\alpha E^\alpha_{ij}} = e^{i\omega\cdot E_{ij}}$$

These quantities are Hermitian operators in our Hilbert space and satisfy

$$\begin{aligned}&[E^\alpha_{ij},E^\beta_{ij}] = if^{\alpha\beta\gamma}E^\gamma_{ij}\\ &[E^\alpha_{ij},\hat{U}_{ij}] = -\lambda^\alpha\hat{U}_{ij}\end{aligned}$$

These are the canonical commutation relations analogous to $[p,x] = i$. The operators corresponding to different links all commute. Homework: verify the minus sign in the second part of the above equation.

The connection between a link and its inverse is

$$\hat{U}_{ij} = \hat{U}^\dagger_{ji}$$

which implies

$$[E^\alpha_{ij}, \hat{U}_{ji}] = \hat{U}_{ji}\lambda^\alpha$$
$$[E^\alpha_{ij}, E^\beta_{ji}] = 0$$

To relate the forward and reversed E's, look at

$$\begin{aligned} R_{ji}(g)|U_{ij}\rangle &= |U_{ij}g^{-1}\rangle \\ &= |(U_{ij}g^{-1}U_{ij}^{-1})U_{ij}\rangle \\ &= R_{ij}(U_{ij}g^{-1}U_{ij}^{-1})|U_{ij}\rangle \end{aligned}$$

This implies for the generators

$$E^\alpha_{ji}|U\rangle = -G(U_{ij})^{\alpha\beta}E^\beta_{ij}|U\rangle$$

where I have introduced the adjoint representation $G(g)^{\alpha\beta}$ by

$$g^{-1}\lambda^\alpha g = G(g)^{\alpha\beta}\lambda^\beta$$

The quadratic casimir operator $E^2_{ij} = \sum_\alpha E^\alpha_{ij}E^\alpha_{ij}$ commutes with R and, by the above relation, does not depend on the direction of the link

$$E^2_{ij} = E^2_{ji}.$$

Now lets go back to the transfer matrix. I want to take the timelike spacing to zero. This means the integrals over g will be dominated for g near the identity. Here the group measure is smooth, and we expand in the group parameters ω using

$$dg \sim \prod_\alpha d\omega^\alpha$$
$$\mathrm{ReTr}g = n - \frac{\omega^2}{4} + O(\omega^4)$$

This gives

$$T = Ke^{-a_0 H + O(a_0^2)}$$

with K an irrelevant normalization and the Hamiltonian

$$H = (g_t/g_s)\left(\frac{g^2}{2a}\sum_{ij}E^2_{ij} + \frac{2}{g^2 a}\sum_p \mathrm{ReTr}\hat{U}_p\right)$$

where I have defined $g^2 = g_t g_s$. The overall g_t/g_s factor goes to unity in the continuum limit. This is the Kogut-Susskind Hamiltonian. The first term represents the electric field squared, while the second, involving spacelike plaquettes, is the magnetic field squared.

This Hamiltonian possesses a large symmetry due to the remaining gauge freedom. The temporal gauge still allows time-independent gauge transformations. An operator that performs such at site i is

$$J_i(g) = \prod_{\{ij\}\supset i} R_i j(g)$$

where the product extends over all bonds eminating from site i. This is a symmetry operator that commutes with H. All physical states should be singlets under this operation in that they satisfy

$$J_i(g)|\psi\rangle = \psi$$

In terms of the generators

$$\sum_{\{ij\}\supset i} E^{\alpha}_{ij}|\psi\rangle = 0$$

The net flux of electric field out of any site should be zero for physical states. This is an exact discrete version of Gauss's law.

Mean field theory

(this section adapted from my rhic97 notes)

Let me consider for simplicity a simple Ising spin system, and I want to study a ferromagnetic transition, giving a magnetization $M = \langle s \rangle$.

Consider a large number of dimensions. Then each site has lots of neighbors, suggesting we might assume their effect can be averaged. Suppose we are in a magnetized state with $\langle s \rangle = M$. Look at one spin in the average field of the others. Calculate its magnetization

$$M = \frac{e^{2dM\beta} - e^{-2dM\beta}}{e^{2dM\beta} + e^{-2dM\beta}} = \tanh(2dM\beta)$$

This can be solved by iterating

$$\begin{aligned} M &\to M + \Delta M \\ \Delta M &= \tanh(2dM\beta) - M \end{aligned}$$

We have a non-trivial fixed point only if

$$\beta \geq \beta_c = \frac{1}{2d}$$

Jensen's inequality provides a more formal approach to mean field theory. Note that e^x is a convex function, $\frac{d^2}{dx^2} e^x = e^x \geq 0$. If x is some stochastic variable, this means that $\langle e^x \rangle > e^{\langle x \rangle}$. Show this for homework.

Consider a "fake" weighting with $P(s_i) = \frac{e^{Hs_i}}{e^H + e^{-H}}$. With this probability distribution

$$\langle s_i \rangle_P = \tanh(H)$$

Thus H might be thought of as a "source" pulling on the spins. Now I manipulate the partition function

$$\begin{aligned} Z &= \sum_{\{s\}} e^{\beta \sum_{\{ij\}} s_i s_j} \\ &= \sum_{\{s\}} e^{\beta \sum_{\{ij\}} s_i s_j - \sum_i \log(P(s_i))} \prod_i P(s_i) \\ &= \langle e^{\beta \sum_{\{ij\}} s_i s_j - H \sum_i s_i + V \log(2\cosh(H))} \rangle_P \\ &\geq \exp(\langle \beta \sum_{\{ij\}} s_i s_j - H \sum_i s_i + V \log(2\cosh(H)) \rangle_P) \\ &= \exp(V(d\beta \tanh^2(H) - H\tanh(H) + \log(2\cosh(H)))) \end{aligned}$$

Thus for any H the true free energy is less than

$$F \leq= F_{mf} = -d\beta \tanh^2(H) + H \tanh(H) - \log(2\cosh(H)).$$

This is an "effective potential" which, depending on the value of β, can have one minimum at $H = 0$ or a double well shape with two minima. The critical point occurs when $\mathrm{O}(H^2)$ term vanishes, *i.e.* the "mass term." This happens at

$$d\beta_c - 1/2 = 0$$

or $\beta_c = \frac{1}{2d}$, as before.

These transitions have been second order, *i.e.* the system evolves continuously with the coupling parameters. The picture is a bit different with three states, where cubic terms can drive us to first order transitions, and physics becomes discontinuous. For example consider the three state Potts model, a system with a Z_3 symmetry. Take $s_i \in \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$. Suppose the bonds have low energy if the spins are "parallel" or equal, higher energy otherwise

$$E = -\sum_{ij} \mathrm{Re} s_i^* s_j$$

Define the magnetization to vanish for a random distribution

$$M = \langle \mathrm{Re} s_i \rangle$$

For mean field theory, we are to replace neighbors with the average and solve self consistently. Now there are two anti-parallel cases, each giving minus half a unit of magnetization

$$M = \frac{e^{d\beta M} - e^{-d\beta M/2}}{e^{d\beta M} + 2e^{-d\beta M/2}}$$

Expand the right hand side for small M

$$M = \frac{d}{2}\beta M + \frac{d}{8}\beta^2 M^2 + \mathrm{O}(M^3)$$

For the Ising case there was no $O(M^2)$ piece, and the $O(M^3)$ piece was negative. Now a non-trivial solution appears before reaching $\beta = 2/d$. The new solution appears discontinuously; when it becomes of lower energy, we have a first order transition! Note that when the extra solution first appears, it is meta-stable and not the lowest energy; one can use the Jensen inequality argument to estimate when it drops below the unmagnetized state. Simulation results show that for three and more dimensions the transition is indeed first order. In two dimensions, however, it is second order and the prediction fails. In one dimension there is no transition, just as for the Ising case.

Mean field theory suggests first order transitions for lattice gauge theory. Lattice gauge theory is summarized by the path integral

$$Z = \sum_U \exp(\beta \sum_P \mathrm{ReTr} U_P)$$

with $U_P = U_1 U_2 U_3 U_4$, and the U_i are link variables running around the plaquette in question. The local symmetry $U_{ij} \to g_i U_{ij} g_j$ implies there is no barrier to twirling a local group of links around. Without gauge fixing U cannot have an expectation value for any β. This is Elitzur's theorem. One should play the Jensen game for more rigor, but proceed naively anyway, trying to find a self consistent expectation for a link. Do Z_2 for simplicity, which gives the same result as the above Ising case except for the replacement $M \to M^3$ for the average field

$$M = \tanh(\beta M^3)$$

Now there is no linear term at all on the right hand side. The prediction is for a strong first order transition. Most lattice gauge transitions in fact are first order: Z_{2-4} in four dimensions; all known gauge groups in 5 or more dimensions. However, in 3-d, the Z_2 gauge model is dual to the Ising model; so, the transition is second order. In 2-d, gauge fixing turns a gauge model into a one dimensional spin system, with no transitions for any finite dimensional group.

(a bit of review) Group integration extracts the "singlet" part of a function in the following sense. Suppose f is a "class function," *i.e.* $f(g) = f(g_1 g g_1^{-1})$. Then I can expand it in traces over the various irreducible representations R of the group

$$f(g) = \sum_R f_R \chi_R(g)$$

where the character $\chi_R(g) = \mathrm{Tr} M_R(g)$ and $M_R(g)$ is the matrix representing g in representation R. These representations include the trivial one, $R = 0$, the fundamental one $R = F$, the adjoint $R = A$, and generally infinitely many more. For irreducible representations, the characters χ satisfy an orthogonality condition

$$\int dg \; \chi_R^*(g) \; \chi_{R'}(g) = \delta_{RR'}$$

From this and $\chi_0(g) = 1$, we see

$$\int dg f(g) = f_0.$$

If we insert a character in some other representation we obtain

$$\int dg \chi_R^*(g) \; f(g) = f_R.$$

This allows us to do some specific integrals integrals quite easily. For example, with $SU(3)$ we have

$$\int_{SU(3)} dg \; (\mathrm{Tr} g)^3 = 1$$

since there is only one singlet in the famous decomposition $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$. This integral lies at the base of the argument below for a first order chiral transition with three massless quark flavors.

So lets apply this to lattice gauge theory. On each bond of our hyper-cubic lattice we have a group element U_{ij}. The Wilson action multiplies these around elementary squares and constructs $U_P = U_1 U_2 U_3 U_4$. The partition function is

$$Z = \int \{dU\} e^{-\beta \sum_P \mathrm{Re}\chi_F(U_P)}$$

Formally, as argued earlier, this represents something like $Z = \mathrm{Tr} e^{aHN_t}$ with a the temporal lattice spacing. The picture, however, is a bit more complicated due to gauge invariance. If we put a group element g_i on each site, we can imagine taking $U_{ij} \to g_i U_{ij} g_j^{-1}$. This change cancels from the action.

In the temporal gauge, all time-like links are set to unity. In this gauge the above transfer matrix argument reduces the path integral to Hamiltonian lattice gauge theory. But on finite periodic temporal lattice, temporal links at a given spatial site form a closed loop. Thus one cannot gauge fix all of them. At each spatial location we must leave one temporal link unfixed, take it to be at time 0. What does integrating over this link correspond to?

In Hamiltonian language, there is an operator $R_i(g)$ that does a local gauge rotation at site i. In particular, for a link to a spatial neighbor

$$R_i(g) U_{ij} R_i^{-1}(g) = g U_{ij}$$

These are all operators in the Hilbert space of the Hamiltonian approach. What the path integral formally reduces to is

$$Z = \mathrm{Tr}\left(e^{-aHN_t} \prod_i (\int dg_i R_i(g_i)) \right)$$

From the above discussion we see that this last integration projects out a gauge singlet. This operator imposes the lattice-gauge-theory version of Gauss's law

$$\delta(\vec{D} \cdot \vec{E}) \leftrightarrow \int dg R_i(g)$$

Now we can generalize and consider not projecting out the singlet everywhere. In particular, at one site I might want to put down a quark-like source. To do this I simply insert the character for the desired representation

$$\int dg \chi_F^*(g) R_i(g)$$

The ratio of the new partition function to the old is the Wilson line or the Polyakov loop. Going back to the path integral, it is just the expectation of a product of temporal links wrapping around the time direction. This Hamiltonian argument explicitly shows how it represents the energy carried by a fixed source in the fundamental representation of the gauge group.

For the quark-less theory, the unfixed temporal links at time 0 have a global symmetry under the center of the gauge group. For the $SU(3)$ of the strong interactions the center is the set $\{1, e^{\pm 2\pi i/3}\}$. By definition, center elements commute with all group elements, and the global change $g_i \to e^{2\pi i/3} g_i$ will cancel out of the temporal plaquettes, each of which involves one g_i and one g_j^{-1}. This is exactly the same symmetry as for the 3 state Potts model, which I argued above should have a first order phase transition. This prediction is well verified by numerical simulation.

The quark kinetic term explicitly breaks this symmetry, so the transition might go away. For massless quarks, it is instead the global chiral symmetry that becomes relevant. This suggests second order for two flavors. For three light flavors the suggestion is first order since there is a quadratic term in the mean field expansion. This arises since the product of three fundamental representations contains a singlet piece, as we well know because three quarks can combine to form a gauge singlet baryon. Generalizing our earlier mean field equation to an $SU(3)$ spin system gives

$$3M = \frac{\int dg\ e^{2d\beta M \mathrm{ReTr} g} \mathrm{Tr} g}{\int dg\ e^{2d\beta M \mathrm{ReTr} g}} = \frac{d\beta M + (d\beta M)^2/2 + \dots}{1 + (d\beta M)^2 \dots} = d\beta M + (d\beta M)^2/2 \dots$$

where I use the earlier $SU(3)$ example integral. The quadratic term means that the solution jumps discontinuously, just as argued earlier for the Potts model. The interpolation between the small and large mass limits for various numbers of flavors is a major area of current study.