Lattice gauge notes

Gauge Fields

What is a gauge theory
- electrodynamics plus isospin
- a theory of phases
- a local symmetry $A_\mu \to g^{-1} A_\mu g + i g_0 g^{-1} \partial_\mu g$
- fields don't transform simply under Lorentz group

Lattice gauge theory closely tied to the middle two definitions. Put a phase on each link. For non-Abelian case, phase becomes an element of the gauge group, i.e. $U_{ij} \in SU(3)$. The analogy is

- $U_{i,i+e_\mu} = e^{iag_0 A_\mu}$
- $U_{ij} = U_{ji}^{-1} = (U_{ij})^\dagger$

Non-trivial gauge field means the curl of the potential is non zero, which in turn means the phase factor around a closed loop is non trivial. The smallest closed path in the lattice is a "plaquette." Consider the phase corresponding to one such

$$U_P = U_{12} U_{23} U_{34} U_{41}$$

In some sense this measures the flux through this plaquette $U_P \sim exp(iea^2 F_{\mu,\nu})$. Thus a nice way to get something like an action is to look at the real part of the trace of this

$$\mathrm{ReTr} U_P \sim \text{const.} - a^4 F_{\mu\nu} F_{\mu\nu} + O(a^6)$$

An overall constant is irrelevant. This suggests to the natural lattice action

$$S(U) = \sum_P \mathrm{ReTr} U_P$$

Now we have our variables and our action; we need to define our path integral as an integral of the exponentiated action over all fields. For a Lie group, there is a natural measure that I will discuss extensively shortly. Using this measure, the path integral is

$$Z = \int (dU) e^{-\beta S}$$

where we obtain the conventional continuum expression if we choose $\beta = 2N/g_0^2$ for group $SU(N)$ and conventionally normalized bare coupling g_0.

Physical correlation functions are obtained as expectation values. Given operator $B(U)$ which depends on the link variables, we have

$$\langle B \rangle = \frac{1}{Z} \int (dU) B(U) e^{-\beta S(U)}$$

Group Integration

For groups there is a natural measure which satisfies several nice properties. Given any function f of the group elements $g \in G$ the integration measure should be invariant under "translation" by an arbitrary fixed element g_1 of the group

$$\int dg\ f(g) = \int dg\ f(g_1 g)$$

An explicit representation for this integration is almost never needed, but fairly straightforward to write down. Suppose a general group element can be is parametrized by some variables $\alpha_1, ...\alpha_n$. For $SU(N)$ there would be $N^2 - 1$ such parameters. Then assume we know some region R for these parameters that covers the group exactly once. Define the n dimensional fully antisymmetric tensor $\epsilon_{\mu_1,...\mu_n}$ such that, say, $\epsilon_{1,2,...n} = 1$. Writing

$$I = A \int_R d\alpha_1...d\alpha_n f(g(\vec{\alpha}))\epsilon_{\mu_1,...\mu_n} \operatorname{Tr}\left((g^{-1}\partial_{\mu_1} g)...(g^{-1}\partial_{\mu_n} g)\right)$$

will have exactly the required invariance properties. Because we have a group $g_1 g(\vec{\alpha}) = g(\vec{\alpha}')$ and we can change variables from α tp α' and the epsilon factor generates exactly the Jacobean needed to absorb this change of variables.

The factor A is at this point arbitrary, but for a compact group it is convenient to normalize the measure so that

$$\int dg\ 1 = 1$$

The left and right measures are equal

$$\int d_r g\ f(g) = \int d_l g_2 \int d_r g_1\ f(g_1 g_2) = \int d_l g\ f(g)$$

Applying this twice shows the measure is unique. For a non-compact group the normalization might be different.

Similarly, we can show

$$\int dg\ f(g) = \int dg\ f(g^{-1})$$

For a discrete group, $\int dg$ is just a sum over the elements. For $U(1) = \{e^{i\theta} | 0 \leq \theta/2\pi\}$

$$\int dg\ f(g) = \int_0^{2\pi} \frac{d\theta}{2\pi} f(e^{i\theta})$$

For $SU(2)$

$$G = \{a_0 + i\vec{a}\cdot\vec{\sigma} | a_0^2 + \vec{a}^2 = 1\}$$

the measure is simply

$$\int dg\ f(g) = \frac{1}{\pi^2}\int d^4a\ f(g)\delta(a^2-1)$$

In particular, $SU(2)$ is exactly a 3-sphere.

- HW: verify the π^2 factor

$SU(3)$: you really never need it, but in case you are curious let me factor a general $SU(3)$ matrix as

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/p_1p_2p_3p_6p_7 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_8 & s_8 \\ 0 & -s_8 & c_8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & p_6 & 0 \\ 0 & 0 & p_7 \end{pmatrix}$$
$$\begin{pmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_5 & s_5 \\ 0 & -s_5 & c_5 \end{pmatrix} \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}$$

Now the angles run over $0 \le \theta_1, \theta_2, \theta_3, \theta_6, \theta_7 < 2\pi$ and $0 \le \theta_4, \theta_5, \theta_8 < \pi/2$. With these limits the integration measure is

$$\int dg\ f(g) = \frac{1}{(2\pi)^5}\int d^8\theta \sin(2\theta_4)\sin(2\theta_5)\sin(2\theta_8) f(g)$$

Note: $SU(3)$ is approximately an $S_5 \times S_3$, but actually it has a slight twist, in mapping an S_5 non-trivially into the group one must cover it 4 times. I.e. we have a generalization of a Moebius strip.

Some integrals are quite easy if we realize that group integration picks out the "singlet" part of a function. Thus

$$\int dg R_{ab}(g) = 0$$

for any irreducible representation other than the trivial one, $R = 1$. One can write

$$\int dg \mathrm{Tr} g\ \mathrm{Tr} g^\dagger = 1$$

$$\int dg \mathrm{Tr} g\ (\mathrm{Tr} g)^3 = 1$$

from the well known $3 \otimes \overline{3} = 1 \oplus 8$ and $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 27$.

My book discusses an algorithm for a general integral, but for here I really only need

$$\int dg\ g_{ij}\ (g^\dagger)_{kl} = I_{ijkl}$$

The group invariance says I can multiply the indices arbitrarily by group element on the left or right. There is only one combination of the indices that can survive

$$I_{ijkl} = \delta_{il}\delta_{jk}/N$$

for $SU(N)$. The normalization is fixed since tracing over ik should give the identity matrix. Another integral that has a fairly simple form is

$$\int dg \; g_{i_1 j_1} \; g_{i_2 j_2} \dots g_{i_N j_N} = 1/N! \epsilon_{i_1 \dots i_N} \epsilon_{j_1 \dots j_N}$$

This is useful for studying baryons.

We now have our full theory. It depends on two parameters, β and K, which correspond to the bare gauge coupling and the quark mass, or equivalently to the lattice spacing and the quark mass. The full action for $SU(N)$ is

$$\begin{aligned} S = &\beta \sum_p (1 - \frac{1}{N} \mathrm{ReTr} U_p) \\ &+ \sum_i \left(\overline{\psi}_i \psi_i - K \sum_\mu \overline{\psi}_i (1 + \gamma_\mu) U_{i,i+e_\mu} \psi_{i+e_\mu} + \overline{\psi}_{i+e_\mu} (1 - \gamma_\mu) U^\dagger_{i,i+e_\mu} \psi_i \right) \end{aligned}$$

Gauge invariance

The lattice gauge action has an exact local symmetry. If I place an arbitrary group element g_i on each site i, the action is unchanged if I replace

$$U_{ij} \to g_i^{-1} U_{ij} g_j$$

One consequence is that any link cannot have a vacuum expectation value

$$\langle U_{ij} \rangle = g_i^{-1} \langle U_{ij} \rangle g_j = 0$$

Integrating over g_j, say, gives zero.

Another consequence is that I can forget to integrate over a tree of links in calculating any gauge invariant observable. An axial gauge uses all links pointing in a given direction.

- HW: solve two dimensional pure Z_2 gauge theory

More general gauges require an analogue of the Fadeev Popov factor. If $B(U)$ is gauge invariant, then

$$\langle B \rangle = \frac{1}{Z} \int d(U) e^{-S} B(U) = \frac{1}{Z} \int d(U) e^{-S} B(U) f(U)/\phi(U)$$

where f is some gauge fixing function and

$$\phi(U) = \int (dg) f(g_i^{-1} U_{ij} g_j)$$

is the integral of f over all gauges.

Wilson loop provides a gauge invariant operator. Lots of things vanish without gauge fixing, i.e. quark and gluon propagators!

Now let me get a bit formal and sloppy. Suppose $f = \delta(h)$ so $\phi = \int (dg)\delta(h)$. The integral of a delta of a function is generically a determinant $\phi^{-1} = \det(\partial h/\partial g)$ which is how you get the usual Fadeev Popov picture.

- Continuum can of worms
- does h completely fix the gauge?
- Gribov copies; etc.?
- lattice sidesteps these issues

Order parameters

Formally we have something like a classical statistical mechanics spin system. The spins U_{ij} are elements of a gauge group G. They are located on the bonds of our lattice. Can this system become "ferromagnetic"? Gauge invariance says $\langle U \rangle = 0$.

- confinement vs free photons?
- $U(1)$ different from $SU(3)$?
- Wilson loop measures force between quark sources
- area law versus perimiter law
- non-local order parameter

With dynamical quarks:

- screening: always perimeter law
- glueballs massive, use mass gap
- massless quarks: no mass gap
- use low temperature Stefan Boltzmann law?

Strong coupling:

- area law from pure gauge theory
- $W \sim (\beta/2N^2)^{\mathrm{area}} = \exp -\mathrm{area} \times \log(2N^2/\beta)$
- mass gap $\langle P_1 P_2 \rangle_c \sim \exp -R \times 4 \log(2N^2/\beta)$
- adjoint loop: perimeter law
- mesons: hopping parameter expansion
- $\langle \phi(x)\phi(0) \rangle \sim (2K)^{2L}$
- finite radius of convergence $K \sim 2$