Progress in relativistic viscous hydrodynamics

Björn Schenke
Physics Department, Brookhaven National Laboratory, Upton, NY

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Workshop on Thermal Photons and Dileptons
BNL

Hydrodynamics

Thermal photon and dilepton production:
Need the temperature $T(\vec{x}, t)$.

Hydrodynamics provides the temperature evolution.

Recent developments:

- 3+1D viscous hydro
- Fluctuating initial conditions

Will also mention status of the equation of state, $\eta/s(T)$.
For bulk viscosity see Kevin Dusling's talk.
For details on photon+dilepton production see Charles Gale's talk.

Hydrodynamics

Hydrodynamics: conservation laws for long wavelength modes

$$\partial_\mu T^{\mu\nu} = 0$$

Generally:

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu}.$$

First order Navier Stokes theory:

$$\pi^{\mu\nu} = \pi^{\mu\nu}_{(1)} = \eta(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \tfrac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha).$$

$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$

Acausal for large k modes, numerically unstable.

Hydrodynamics

Second order theory:

$$\pi^{\mu\nu} = \pi^{\mu\nu}_{(1)} + \text{second derivatives.}$$

Most general expression to second order in gradients in flat space for a conformal theory is

$$\pi^{\mu\nu} = \pi^{\mu\nu}_{(1)} - \tau_\pi \left[\Delta^\mu_\alpha \Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}(\nabla_\alpha u^\alpha) \right]$$
$$- \frac{\lambda_1}{2\eta^2}\pi^{<\mu}{}_\lambda \pi^{\nu>\lambda} - \frac{\lambda_2}{2\eta}\pi^{<\mu}{}_\lambda \Omega^{\nu>\lambda} - \frac{\lambda_3}{2}\Omega^{<\mu}_\lambda \Omega^{\nu>\lambda}$$

τ_π, κ, $\lambda_1, \lambda_2, \lambda_3$: five independent second order transport coefficients.

Baier, Romatschke, Son, Starinets, Stephanov, JHEP 0804, 100 (2008)

Hydrodynamics

Using the set of equations

$$\partial_\mu T^{\mu\nu} = 0$$

and

$$\pi^{\mu\nu} = \pi^{\mu\nu}_{(1)} - \tau_\pi \left(\tfrac{4}{3} \pi^{\mu\nu} \partial_\alpha u^\alpha + \Delta^\mu_\alpha \Delta^\nu_\beta u^\sigma \partial_\sigma \pi^{\alpha\beta} \right)$$

is now standard (some also include vorticity).

When bulk viscosity is included (non-conformal fluid)

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu} - \Pi \Delta^{\mu\nu}$$

see B. Betz, D. Henkel, D. Rischke, Prog.Part.Nucl.Phys.62, 556-561 (2009) for structure of bulk terms

QCD enters through the equation of state: $P = P(\epsilon)$
(from e.g. lattice QCD / hadron gas model)

Describing heavy-ion collisions with hydro

- Input: initial conditions
  - Glauber model
  - Color glass condensate
- Input: Equation of state
  - First order phase transition (bag model)
  - Lattice QCD + hadron gas models
- Input: Free parameters
  - Use one set of experimental data to fix
    e.g. $\left.\frac{dN}{dy p_T dp_T}\right|_{b=0\,\mathrm{fm}}$ and $\frac{dN}{dy}(b)$

    Example parameters at RHIC:
    $\varepsilon_{0,\mathrm{max}} \approx 30\,\mathrm{GeV/fm}^3$, $\tau_0 \approx 0.6\,\mathrm{fm}/c$, $T_{\mathrm{fo}} \approx 130\,\mathrm{MeV}$
- Output: Predict other observables:
  - flow, HBT, thermal photons and dileptons, ...

3+1 dimensional viscous hydrodynamics

3+1D relativistic viscous hydrodynamic simulations are now available:

MUSIC: MUScl for Ion Collisions:

MUSCL = Monotonic Upstream Centered Scheme for Conservation Laws

B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

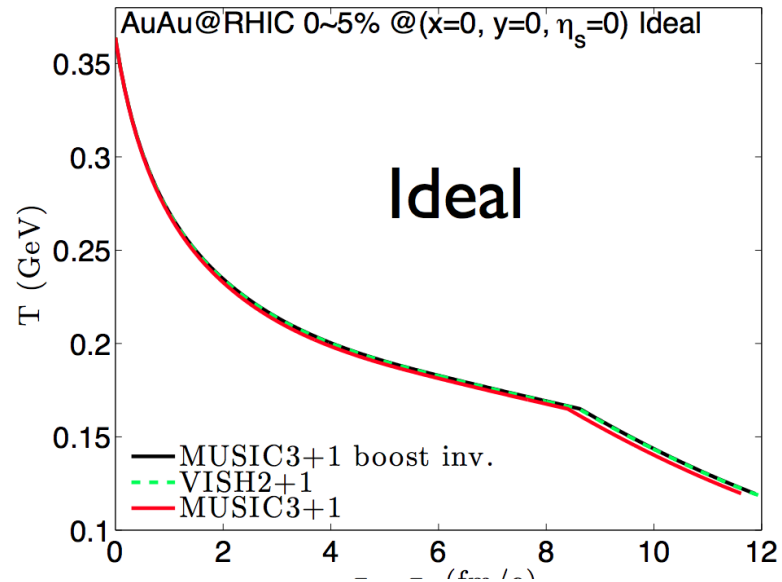
- Use the Kurganov-Tadmor (KT) algorithm
  - low numerical viscosity
  - ability to deal with shocks

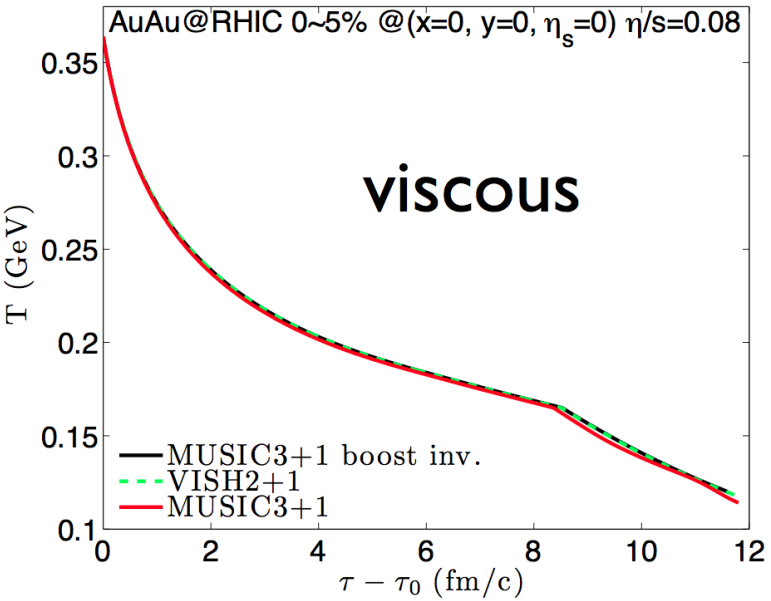
  A. Kurganov, E. Tadmor, Journal of Computational Physics **160**, 241-282 (2000)
- 3+1 dimensions and $\tau - \eta$ coordinates
- Cooper-Frye freeze-out
  with sophisticated freeze-out surface construction
- Includes different equations of state including s95p-v1 from Huovinen and Petreczky (Lattice-QCD)

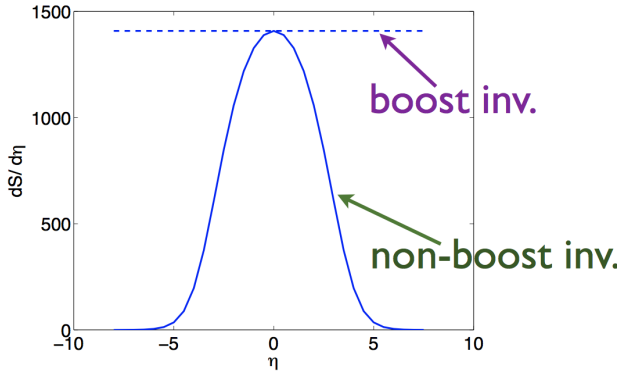
  P. Huovinen and P. Petreczky, Nucl. Phys. **A837**, 26-53 (2010)

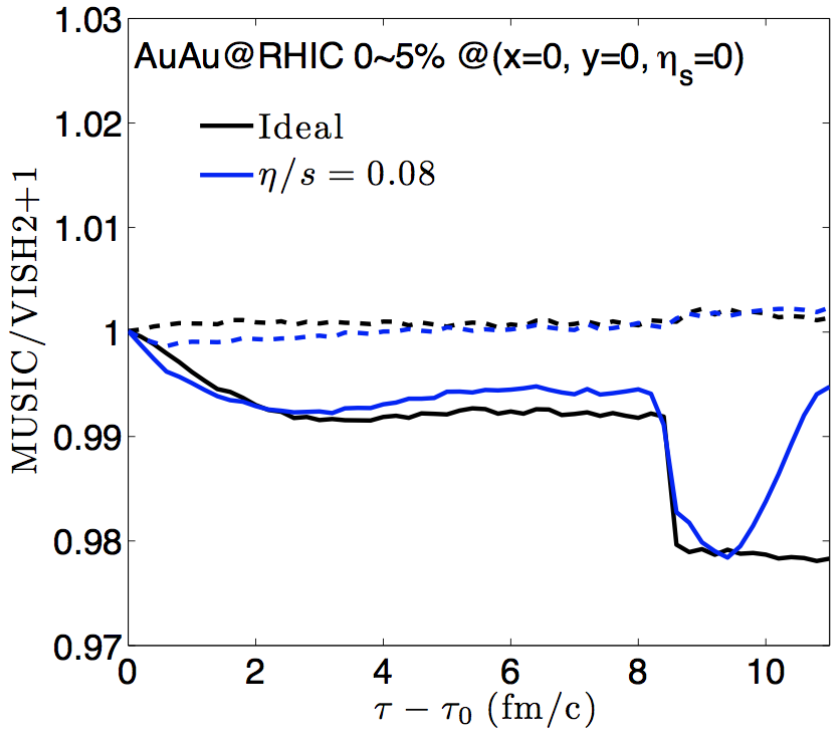
3+1 dimensional viscous hydrodynamics

What changes? Midrapidity: Very good agreement with 2+1D codes.





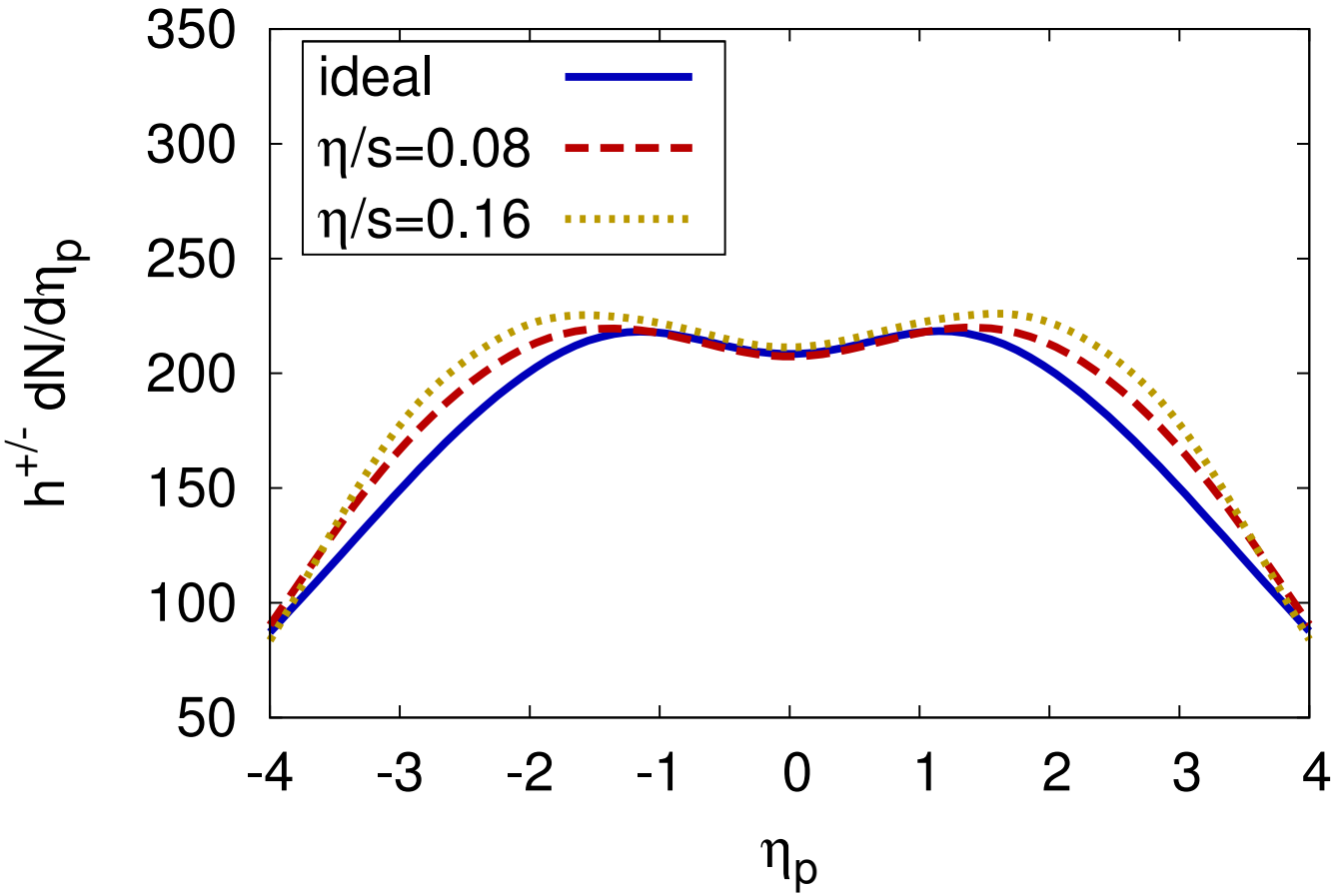




C. Shen, B. Schenke, U. Heinz

3+1 dimensional viscous hydrodynamics

Viscosity has very interesting and important effect on the longitudinal dynamics:



B. Schenke, S. Jeon, and C. Gale, arXiv:1109.6289 (2011)

confirmed also in recent work by P. Bozek, arXiv:1110.6742

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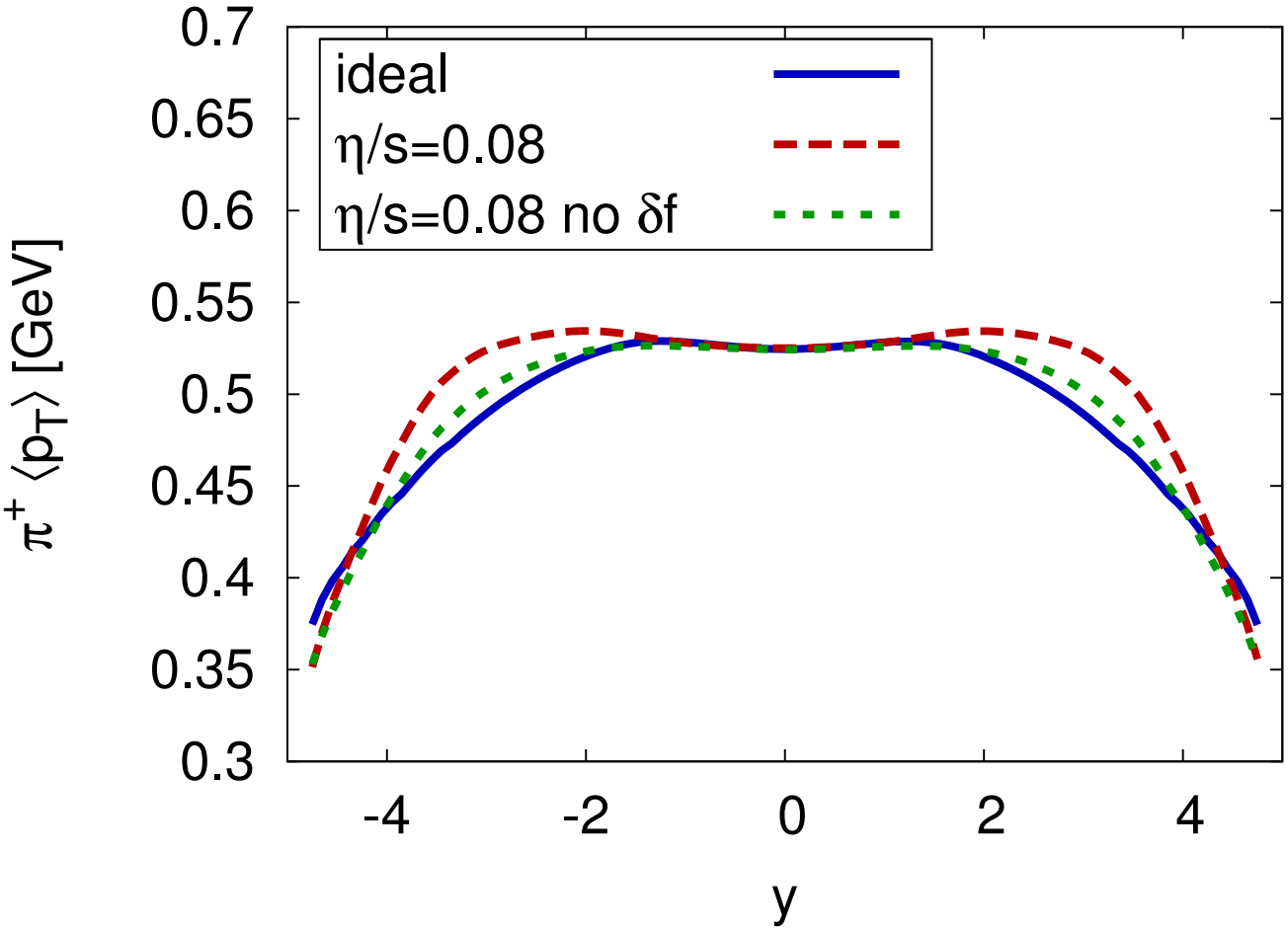

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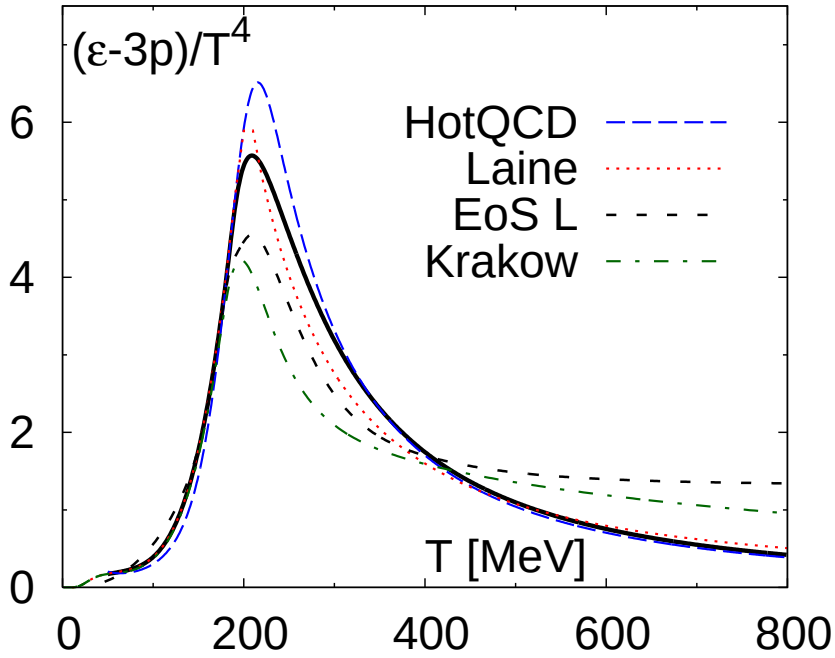
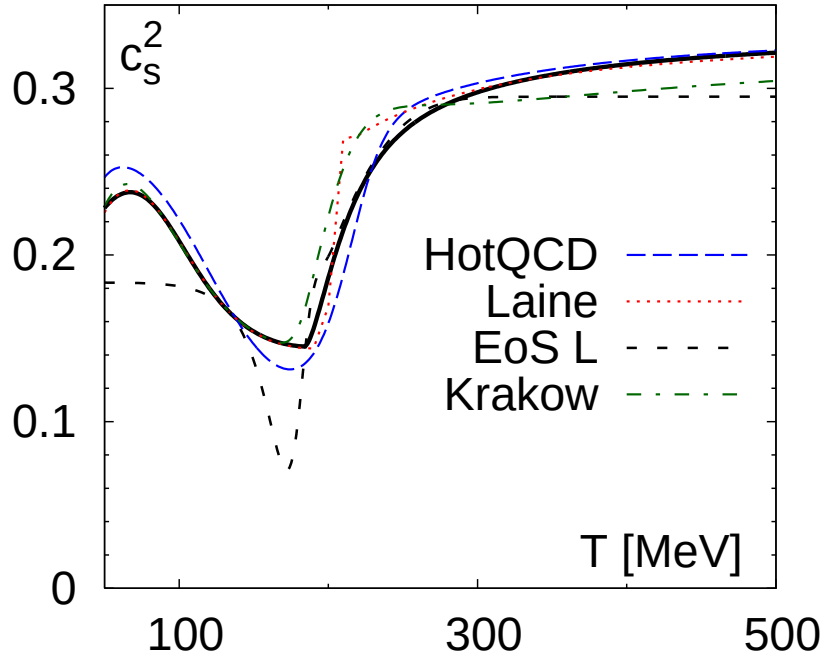
Equation of state

Comparison of different equations of state in hydrodynamic evolution:

P. Huovinen, P. Petreczky, Nucl.Phys.A837:26-53 (2010)





Solid black: Parametrization from P. Huovinen, P. Petreczky, Nucl.Phys.A837:26-53 (2010)

HotQCD: HotQCD collaboration, Phys.Rev.D80:014504 (2009)

Laine: M. Laine and Y. Schröder, Phys. Rev. D73, 085009 (2006)

EoS L: H. Song and U. W. Heinz, Phys. Rev. C 78, 024902 (2008) using Wuppertal-Budapest results

Krakow: M. Chojnacki et al, Acta Phys. Polon. B 38, 3249 (2007) and Phys. Rev. C 78, 014905 (2008)

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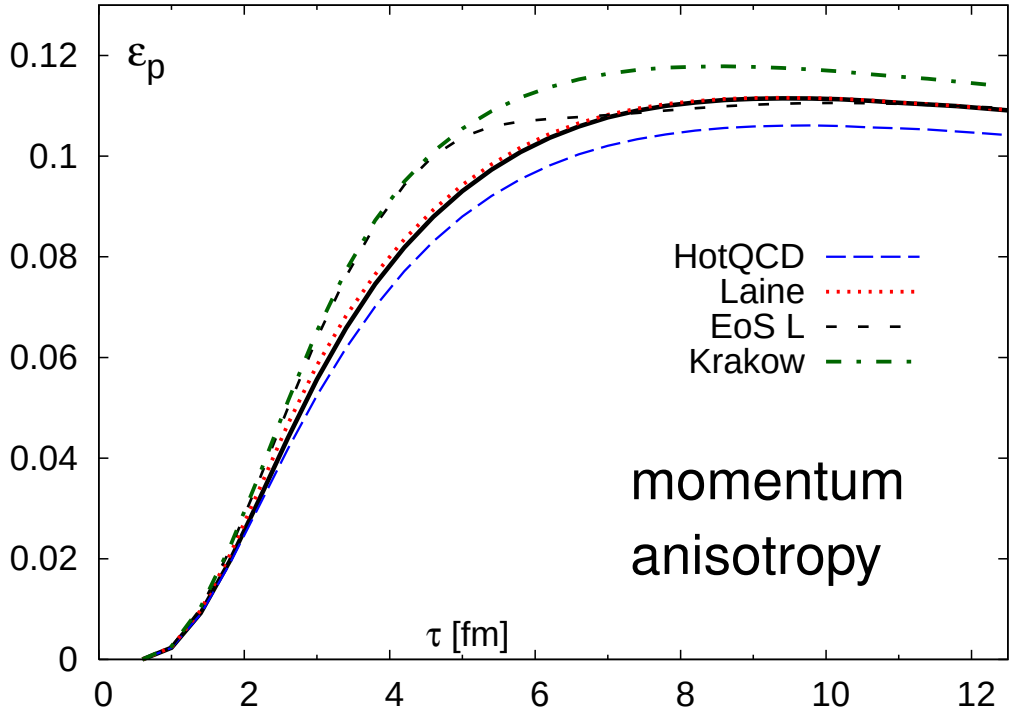

ideal hydro

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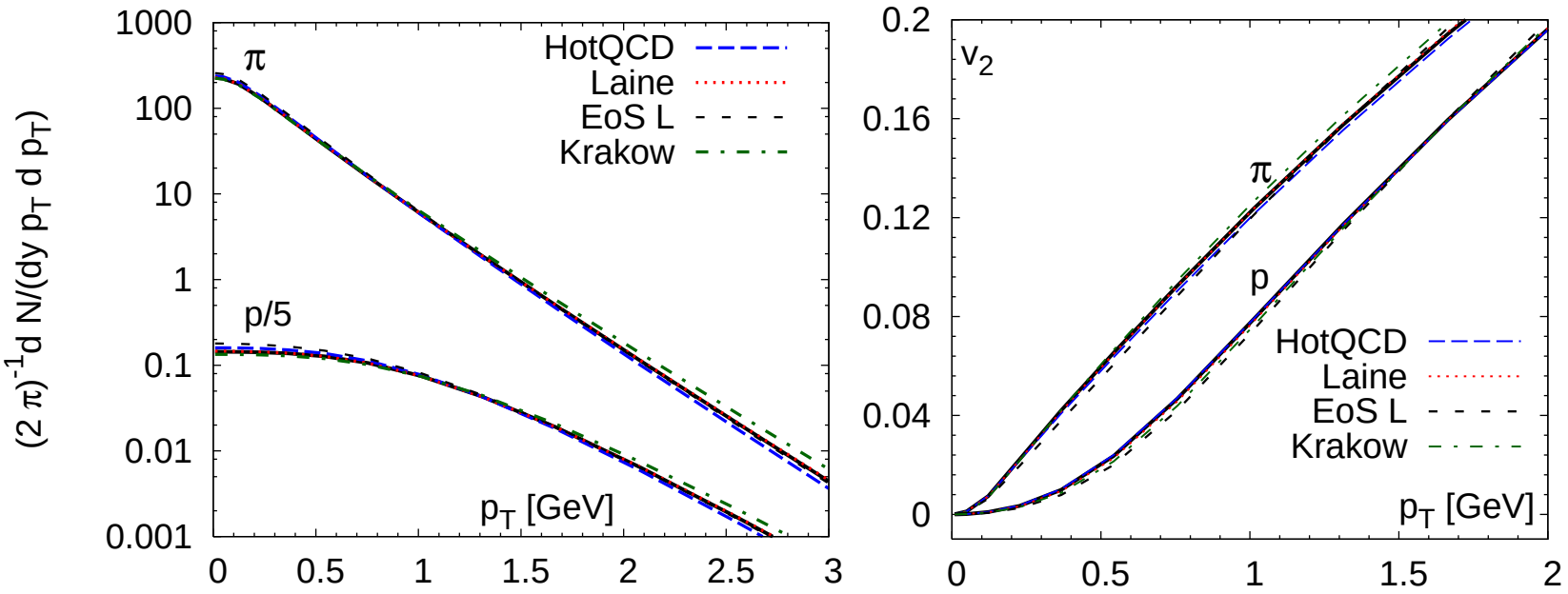

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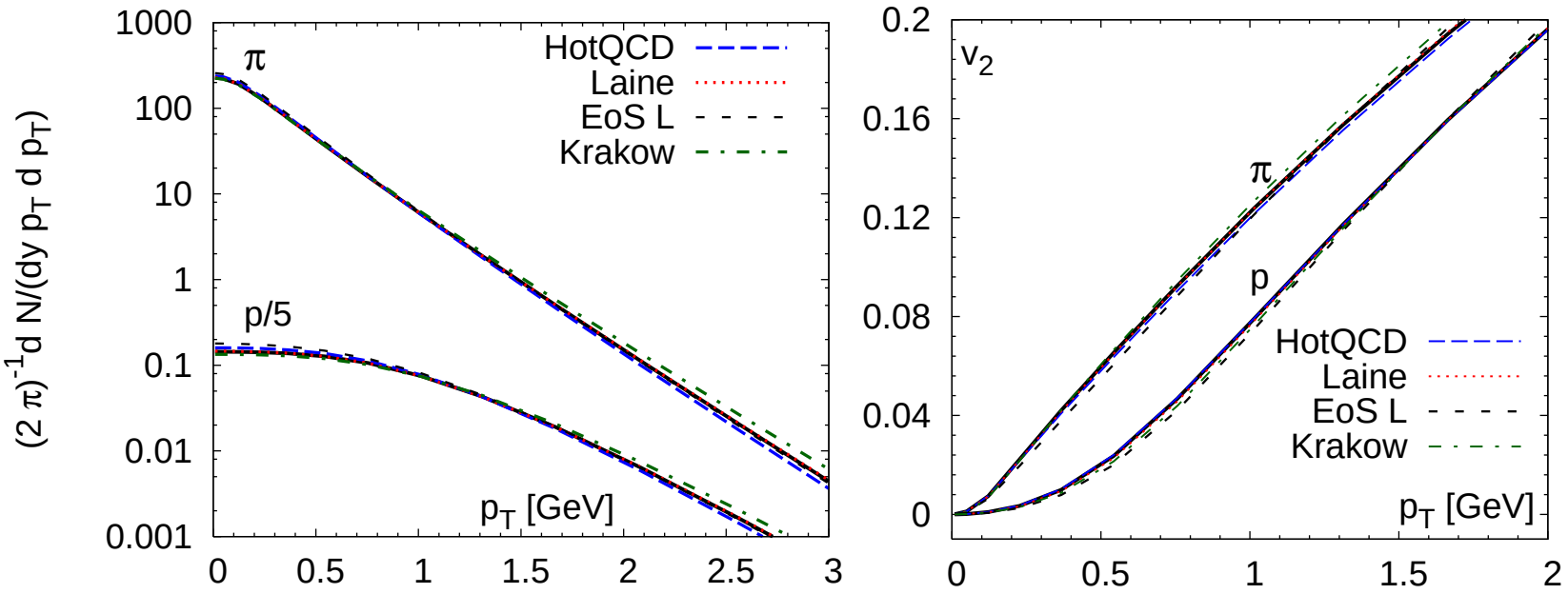


"Differences in the lattice EoS parametrization in the literature are small and not observable in the p_T-differential elliptic flow."

Possibly different for photons and dileptons! (see talk by C. Gale)

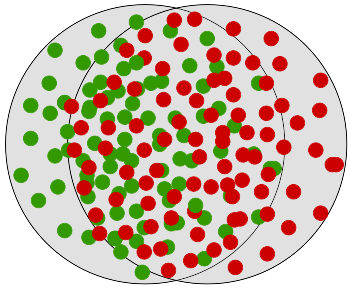
Another recent lattice equation of state:

S. Borsanyi et al, JHEP 1011:077 (2010)

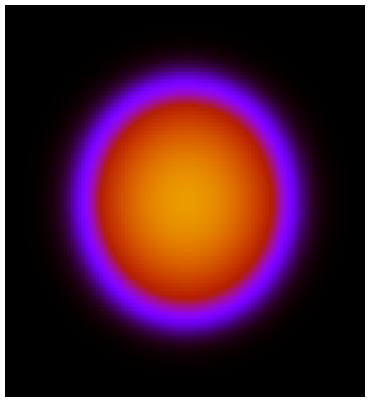
Wuppertal-Budapest lattice results - used in V. Roy and A.K. Chaudhuri, arXiv:1103.2870

Event-by-event viscous hydrodynamics

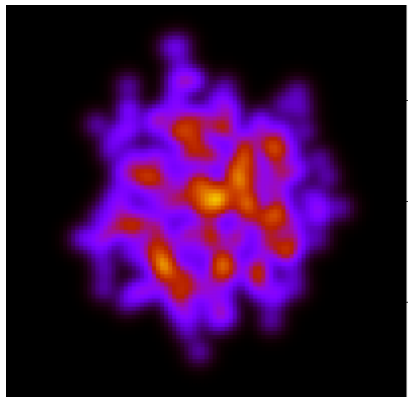
average, then evolve



evolve, then average

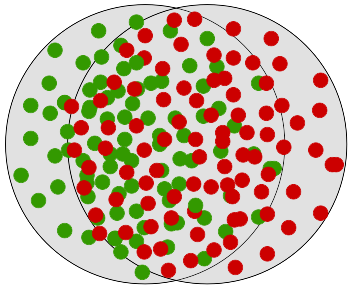


initial energy density

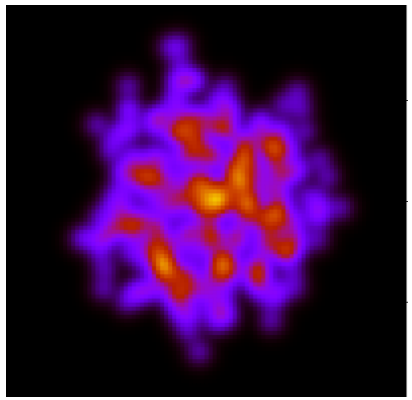


initial energy density

Event-by-event viscous hydrodynamics



evolve, then average

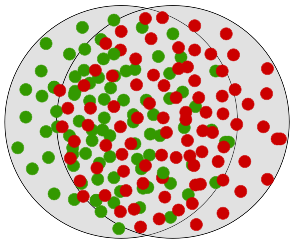


initial energy density

A simple event-by-event initialization

Initialization:

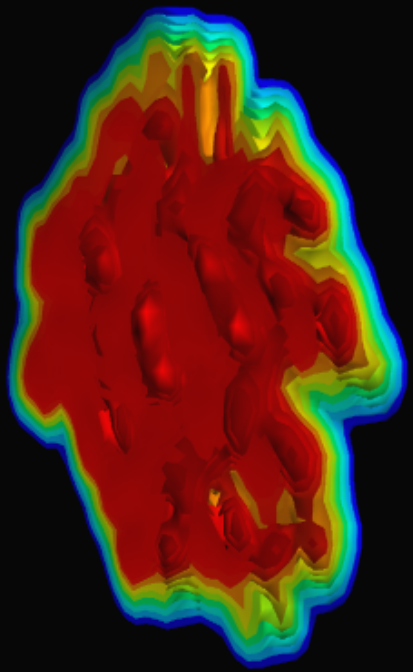
- Sample Woods-Saxon distributions
  to determine all nucleon positions
- Overlap those distributions using impact parameter b



b is sampled from $P(b)db = 2bdb/(b^2_{\rm max} - b^2_{\rm min})$

- Nucleon-nucleon collision occurs if distance is $< \sqrt{\sigma_{NN}/\pi}$
- At position of collision add 2D-Gaussian energy density distribution with width σ_0.
  For now we use $\sigma_0 = 0.4\,\mathrm{fm}$.

Single event initial conditions



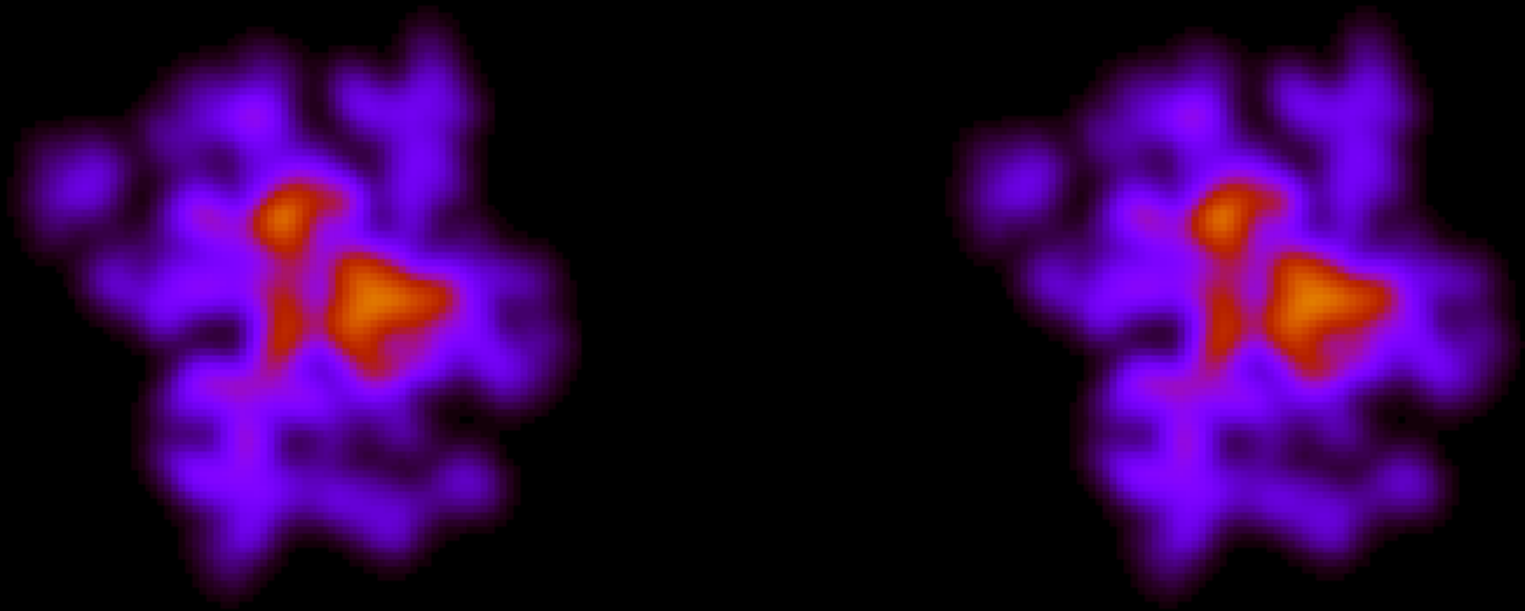
beam direction

contours of constant temperature

Viscosity in a single event

ideal

$\eta/s = 0.16$

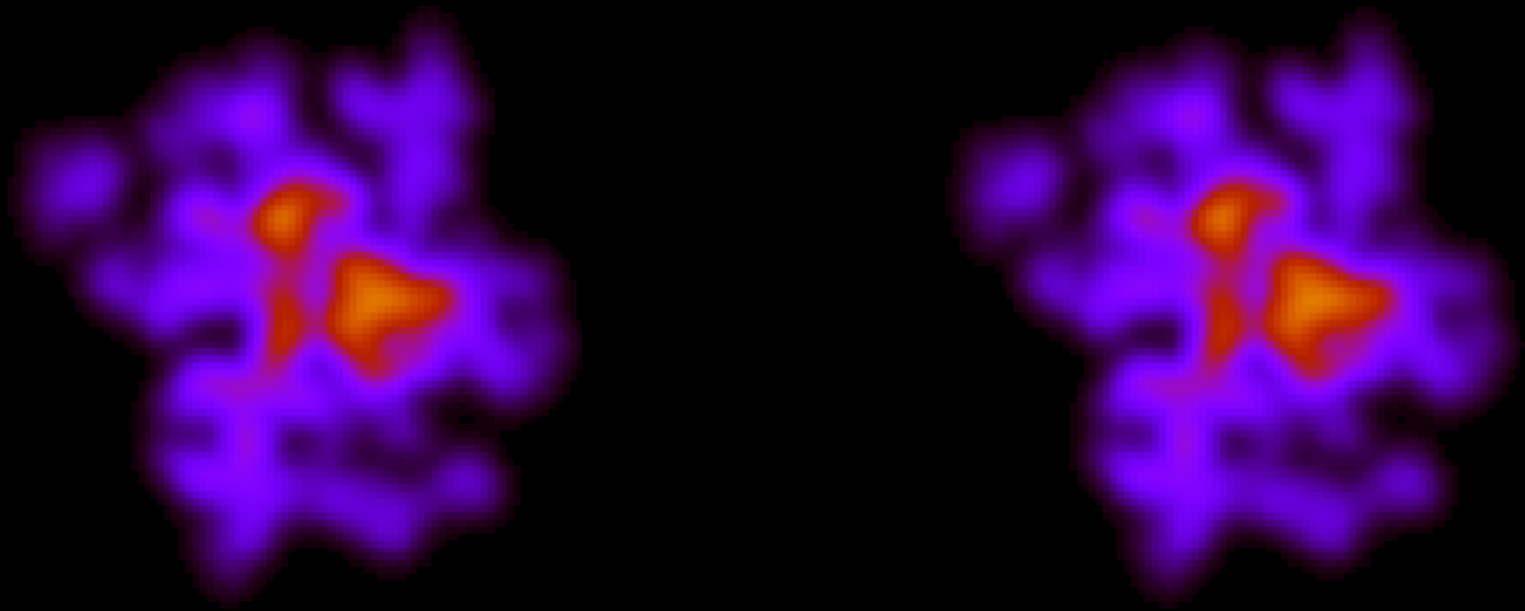


energy density (scale adjusted with time)

Viscosity in a single event

ideal

$\eta/s = 0.16$



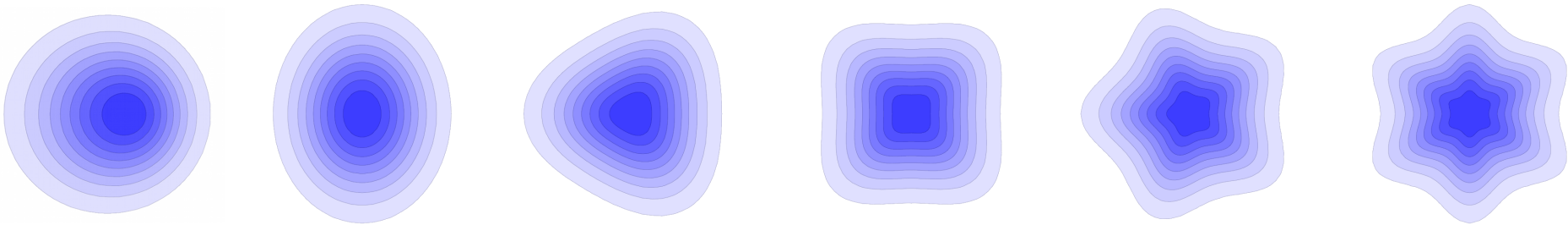
energy density (scale adjusted with time)

Higher harmonics

Use event-by-event hydro to compute higher flow harmonics, get better control over initial conditions and transport properties.

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_n (2v_n \cos(n\phi))\right)$$

When including fluctuations, all moments appear:



$n = 1$ $n = 2$ $n = 3$ $n = 4$ $n = 5$ $n = 6$

... and $n > 6$.

Event plane

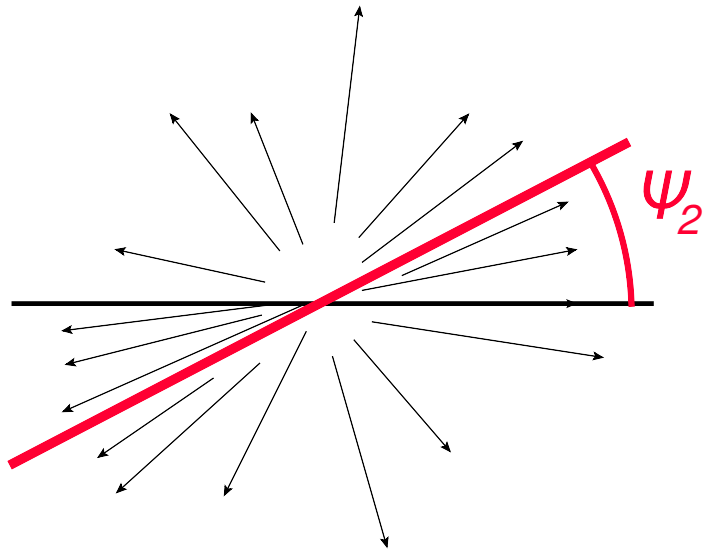
To get non-zero odd moments, we rotate the event plane in each event.

Event plane is defined by the angle:

$$\psi_n = \frac{1}{n} \arctan \frac{\langle p_T \sin(n\phi) \rangle}{\langle p_T \cos(n\phi) \rangle}$$

using particle momenta.

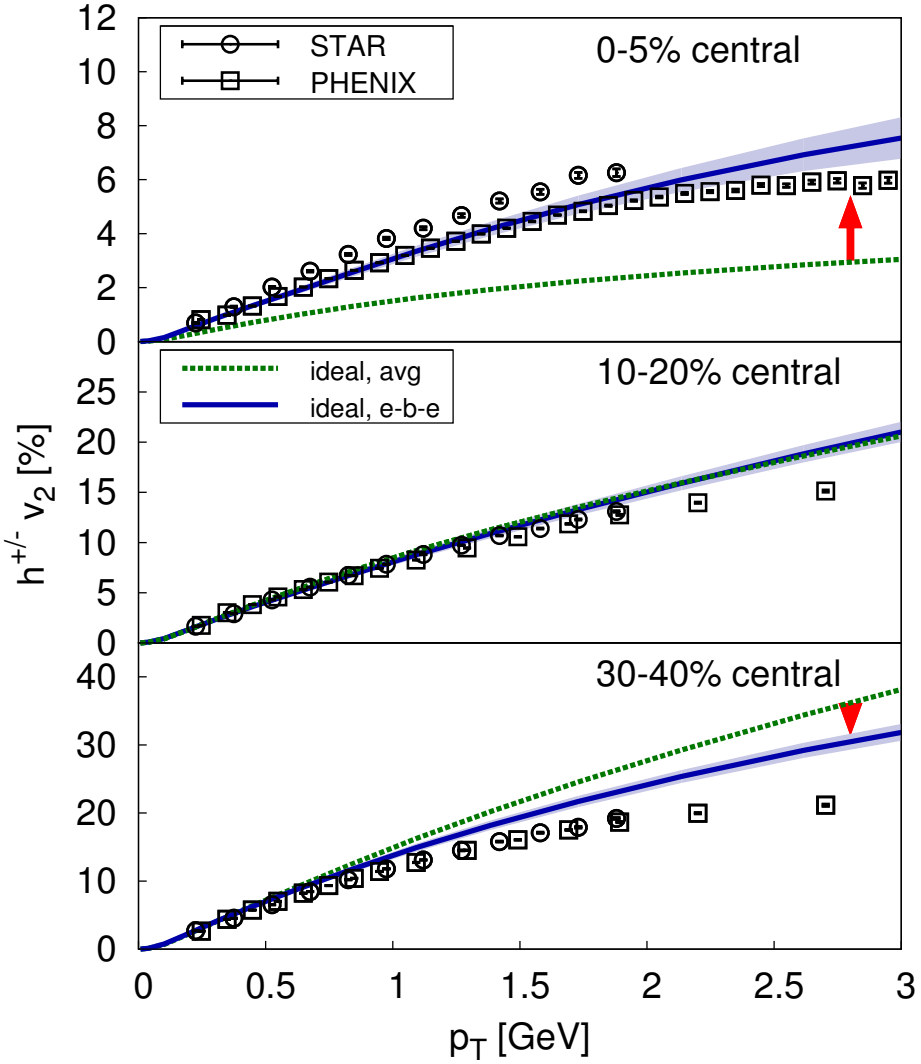
A.Poskanzer and S.Voloshin, Phys.Rev.C58:1671-1678 (1998)



$$v_n = \langle \cos(n(\phi - \psi_n)) \rangle$$

... different angle for every flow coefficient.

Flow results from e-b-e viscous MUSIC



Most central collisions:
fluctuations increase elliptic flow

Larger centralities:
fluctuations decrease elliptic flow

Experimental data: J. Adams et al. (STAR), Phys.Rev.C72, 014904 (2005)
A. Adare et al. (PHENIX), Phys.Rev.Lett.105, 062301 (2010)

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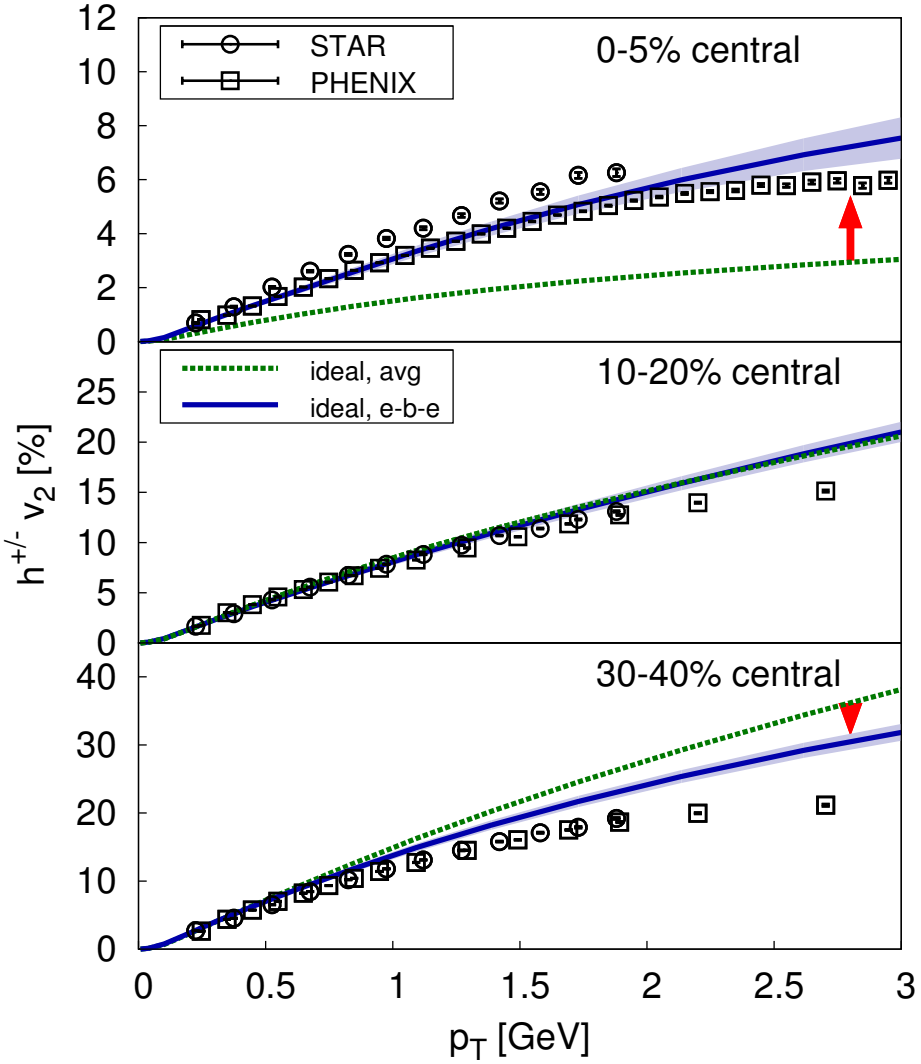
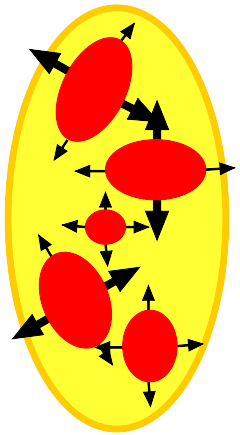


Why?

- Measure with respect to the event-plane:
  Maximizes measured flow.
- Competing effect:
  Substructure of hotspots



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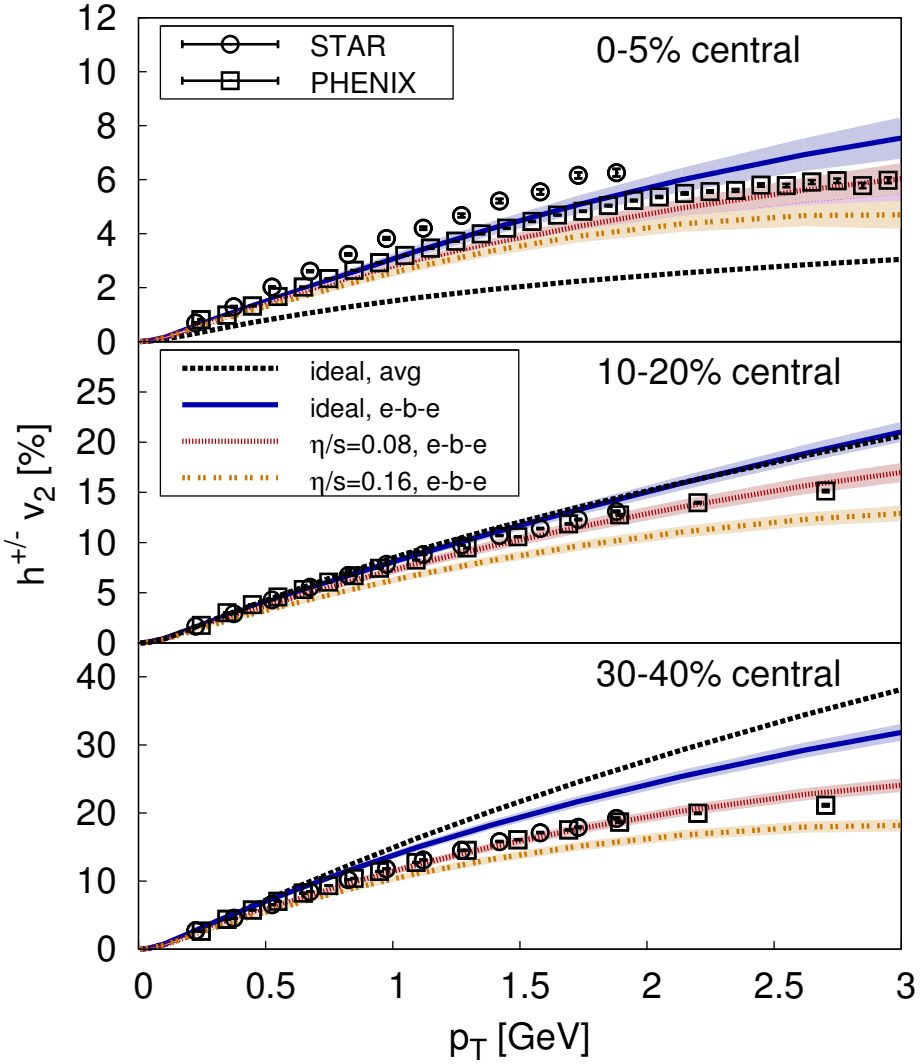


B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

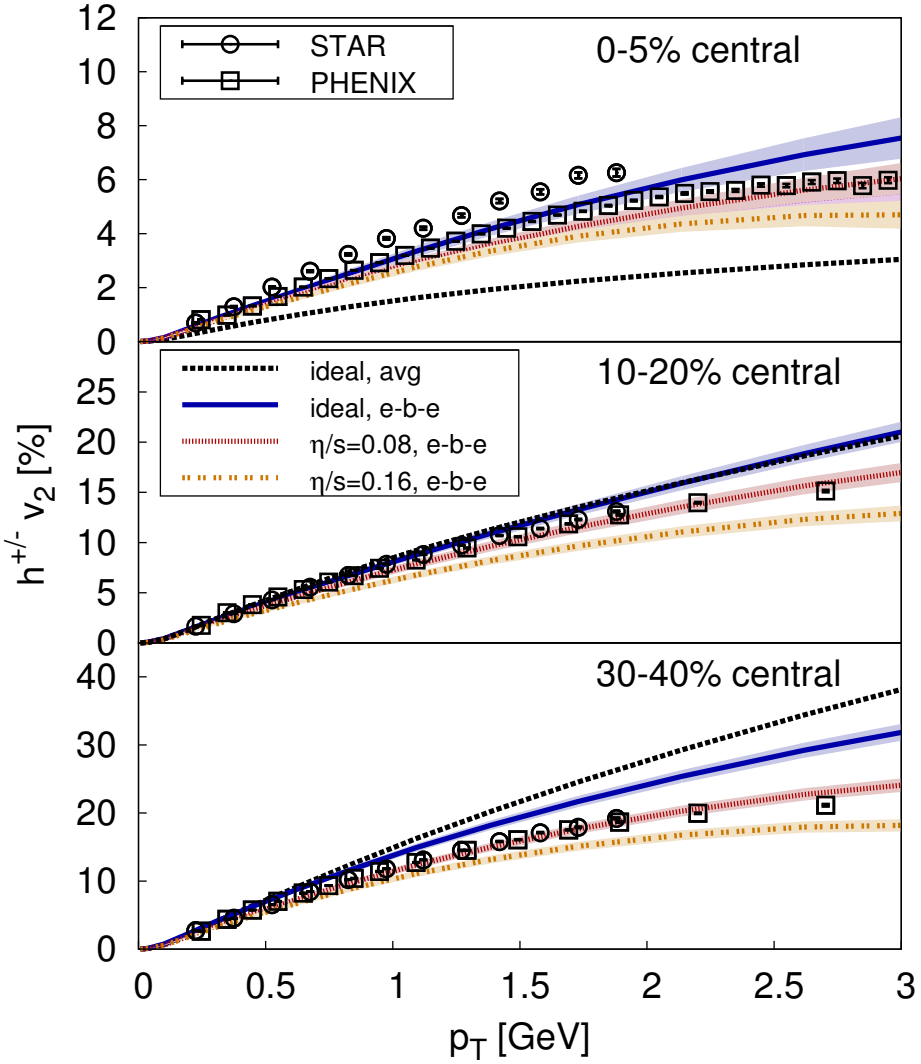
Now add shear viscosity.

$\eta/s = 0.08$ and

$\eta/s = 0.16$

Experimental data: J. Adams et al. (STAR), Phys.Rev.C72, 014904 (2005)
A. Adare et al. (PHENIX), Phys.Rev.Lett.105, 062301 (2010)

Flow results from e-b-e viscous MUSIC



B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

- event-by-event fluctuations important!
- Viscosity is **very low**.
  The lower bound of viscosity/entropy density conjectured from AdS/CFT duality is
  $\eta/s = 1/4\pi \approx 0.08$
  (red curves)

Experimental data: J. Adams et al. (STAR), Phys.Rev.C72, 014904 (2005)
A. Adare et al. (PHENIX), Phys.Rev.Lett.105, 062301 (2010)

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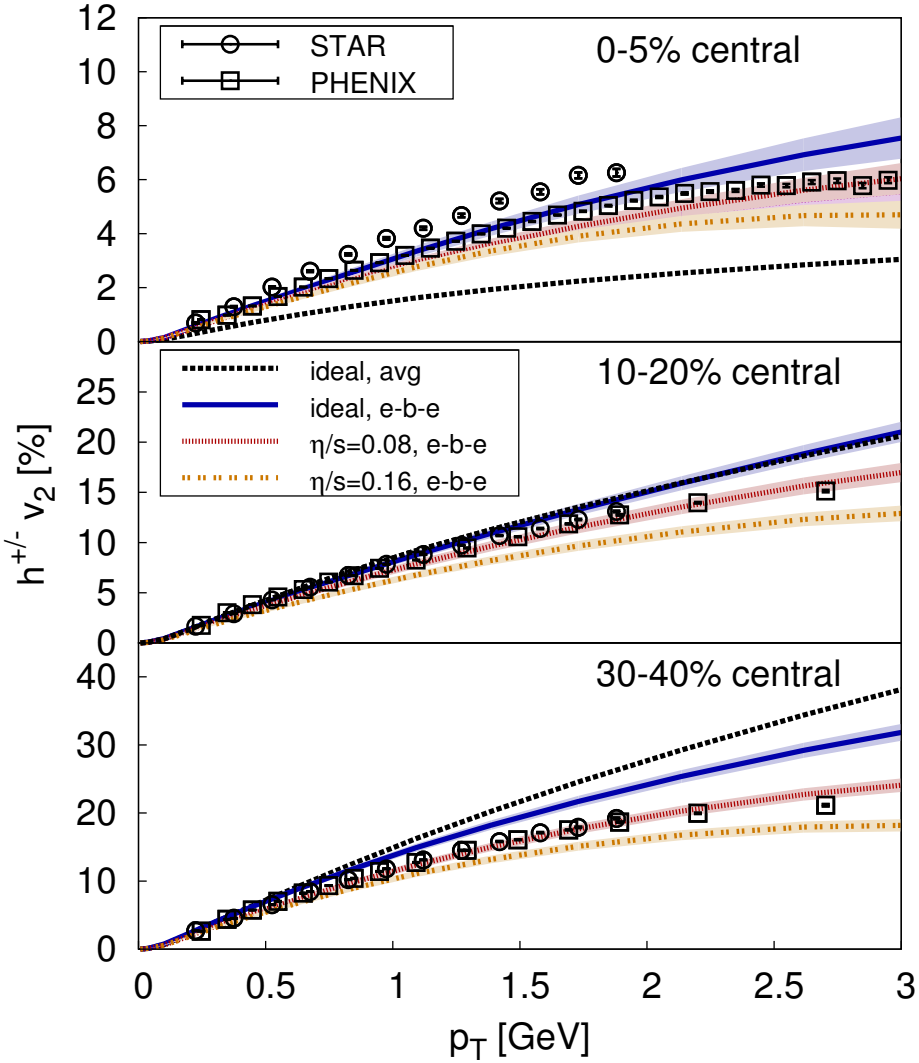

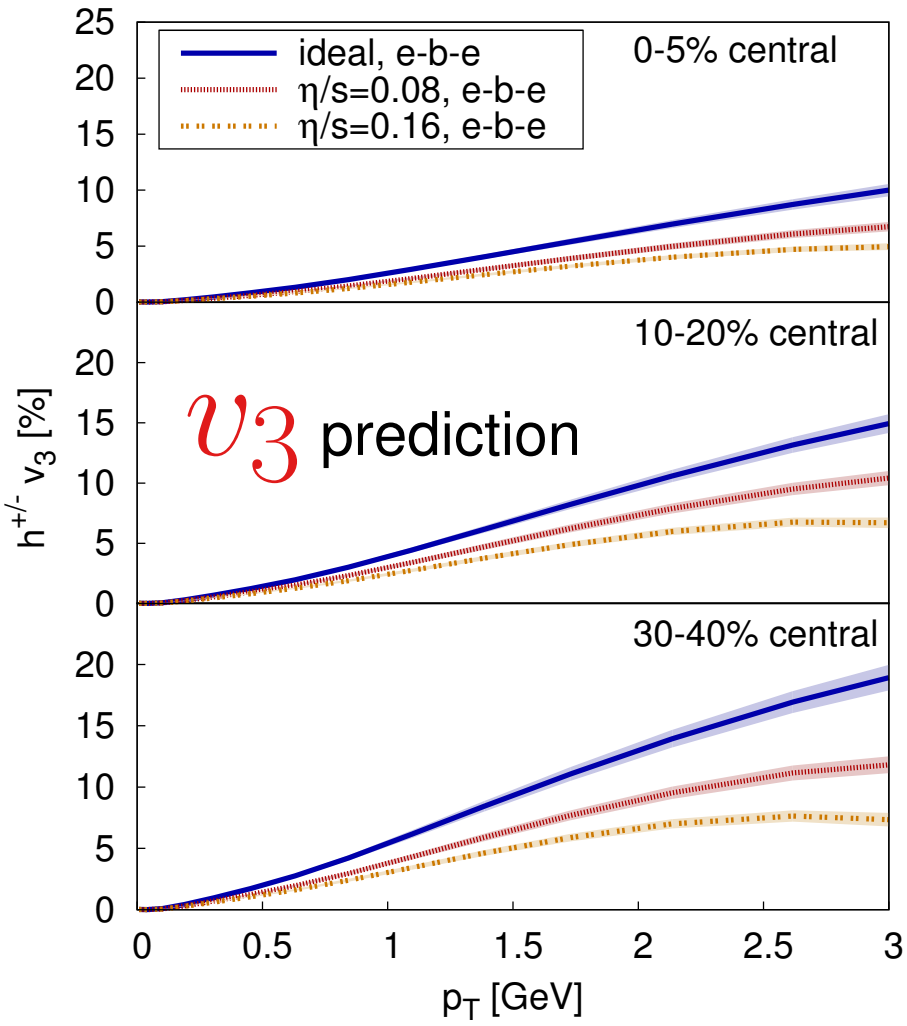

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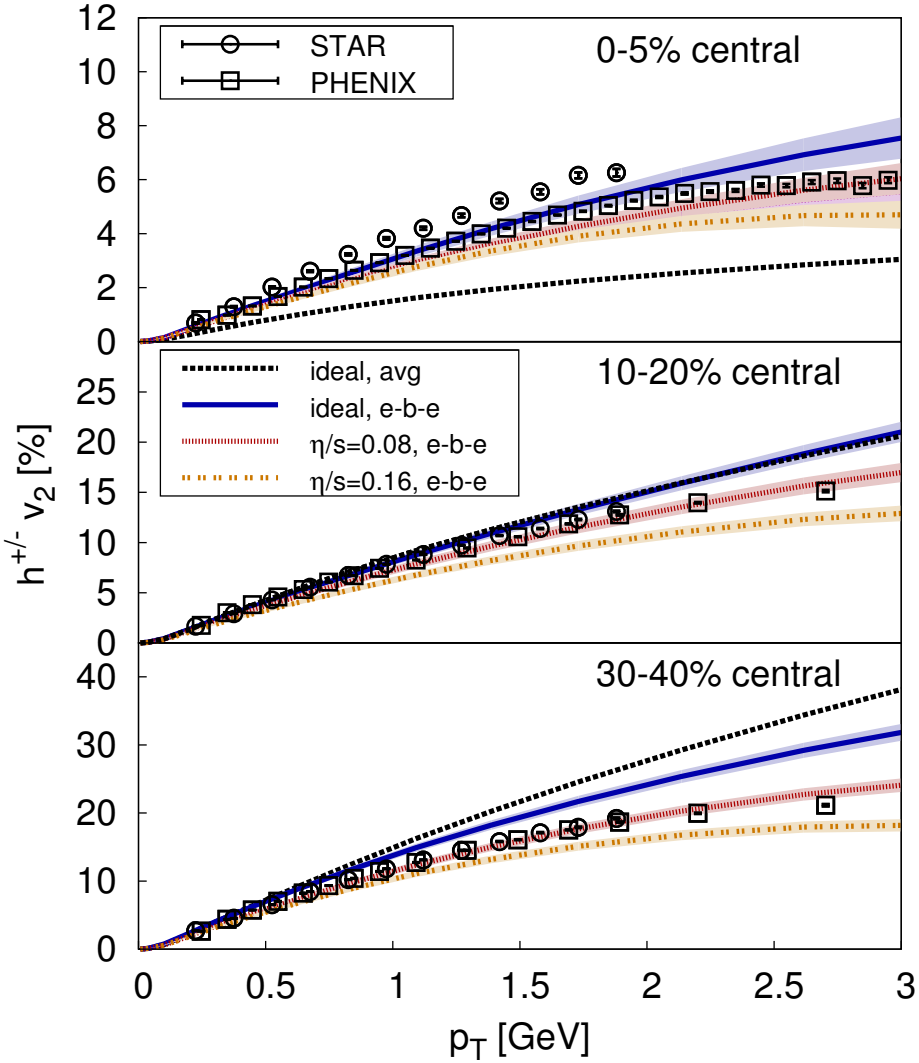

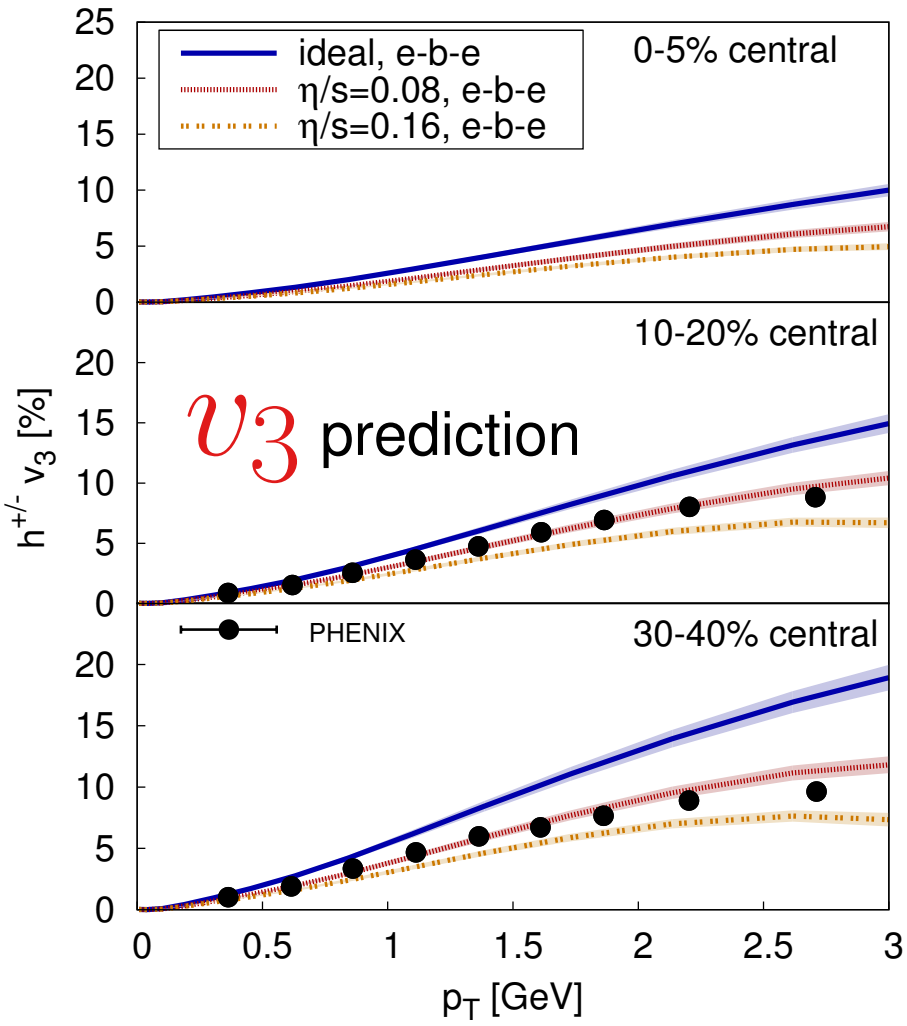

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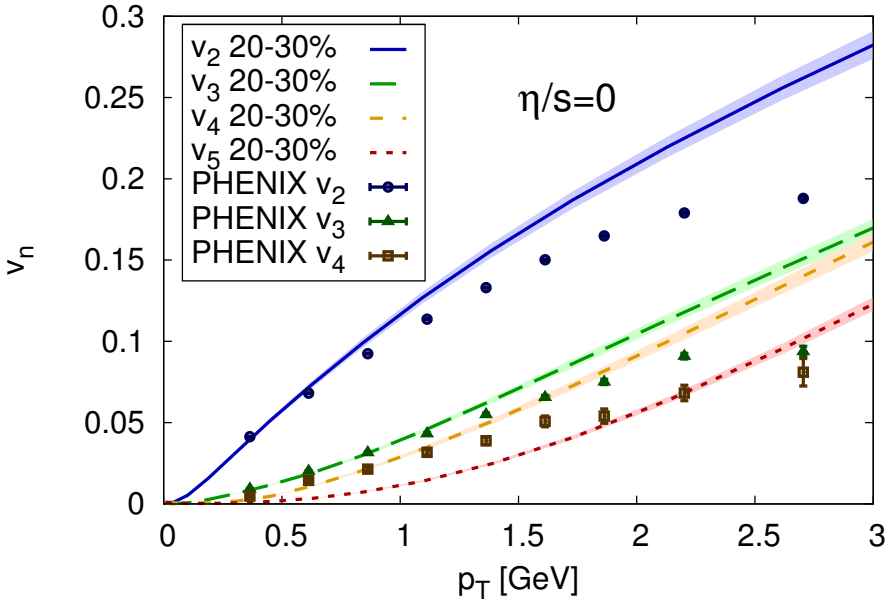
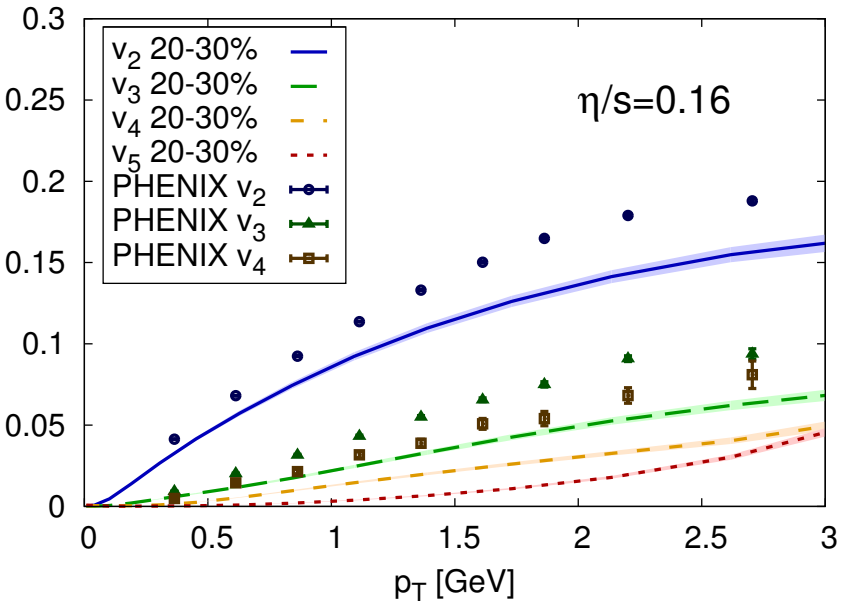
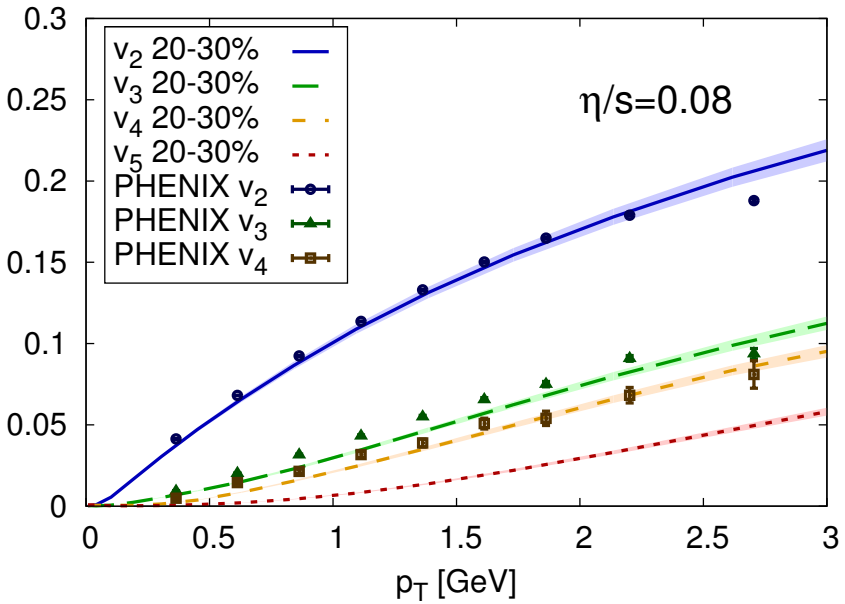
A. Adare et al. (PHENIX), Phys.Rev.Lett.105, 062301 (2010) , A. Adare et al. (PHENIX), arXiv:1105.3928 (2011)

Using higher harmonics to determine η/s





Glauber initial conditions

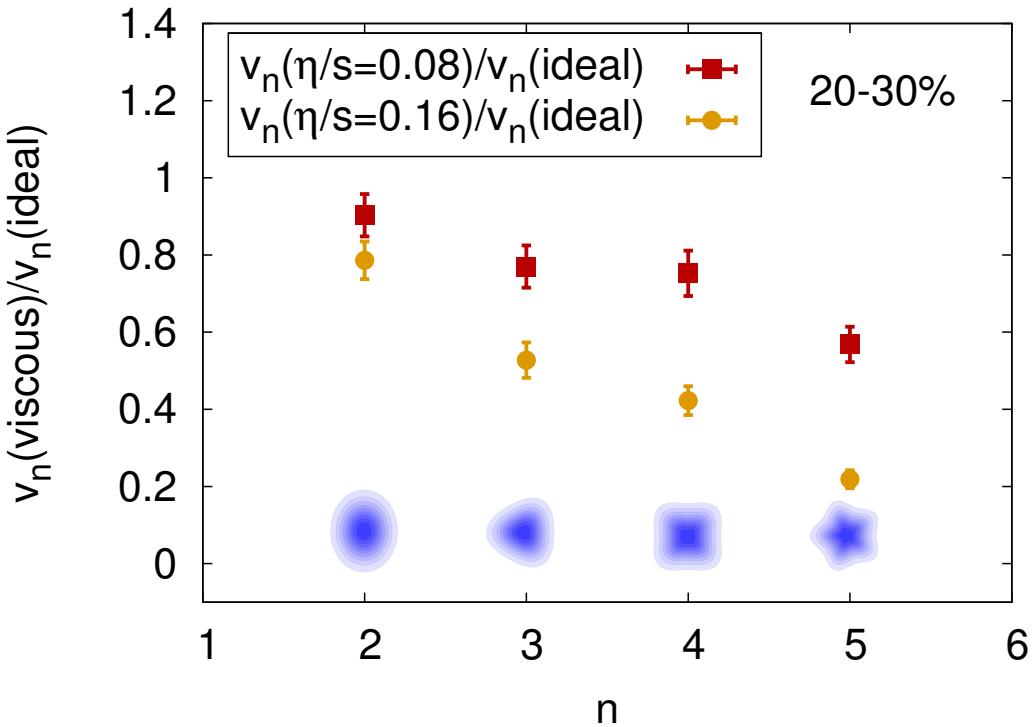


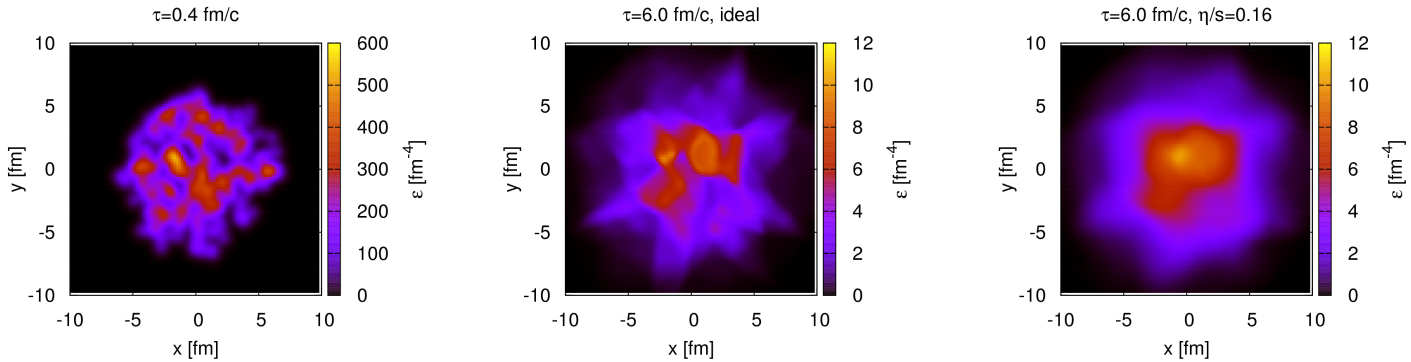
This is promising.
Need systematic study of all v_n as function of initial conditions, granularity, η/s, ...

Experimental data: PHENIX, arXiv:1105.3928

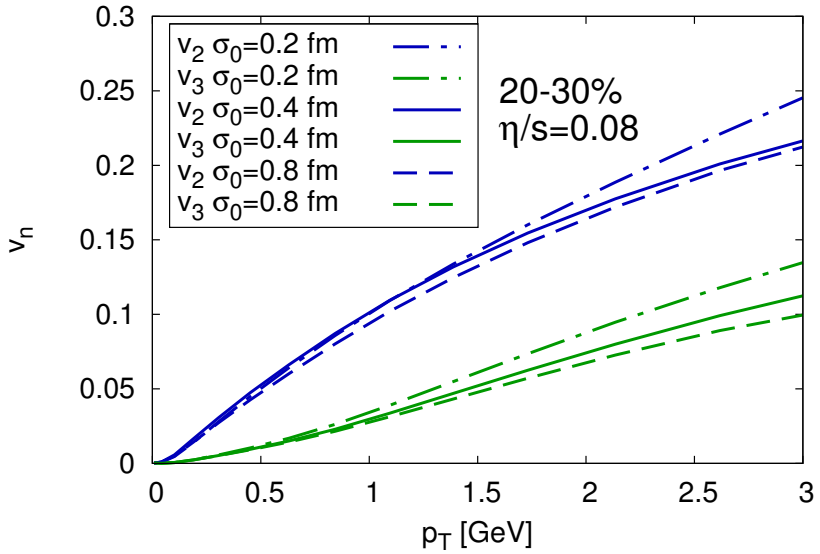
Dependence on η/s

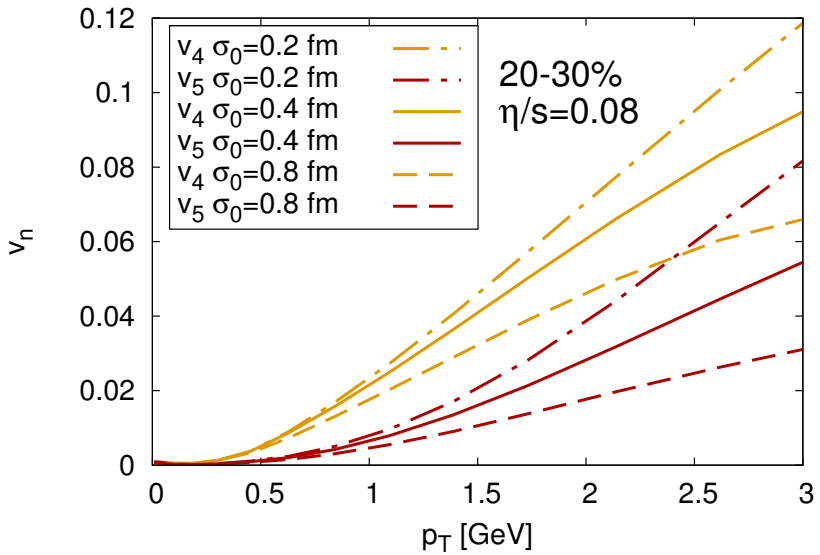
Higher Fourier coefficients are suppressed more by viscosity.

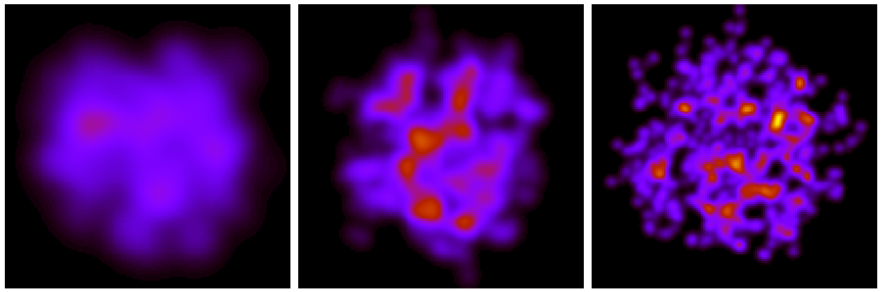




Dependence on initial granularity







$\sigma_0 = 0.8\,\mathrm{fm}$ $0.4\,\mathrm{fm}$ $0.2\,\mathrm{fm}$

v_2: weak dependence

Higher harmonics:
much stronger dependence

However, increase due to larger granularity is unlikely to compensate large suppression for $\eta/s = 0.16$.

Effect of visc. corr. to distribution function δf

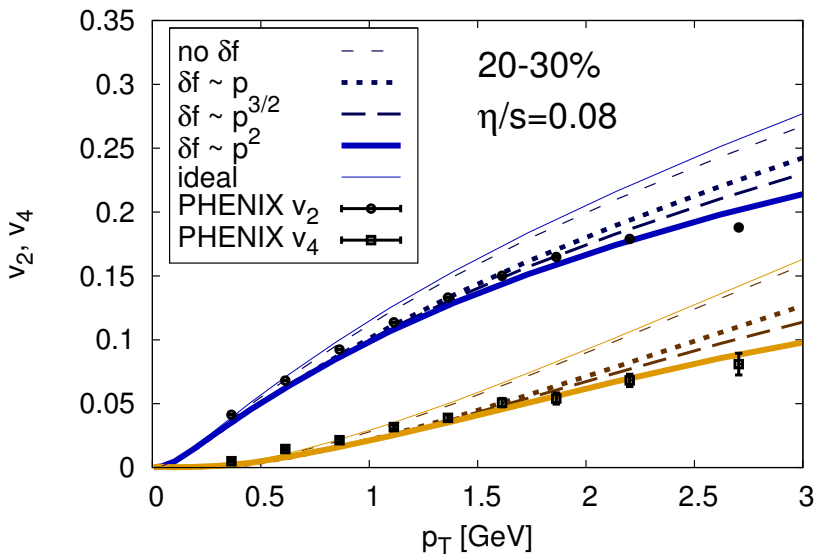
Cooper Frye + viscosity: $f \to f + \delta f$:

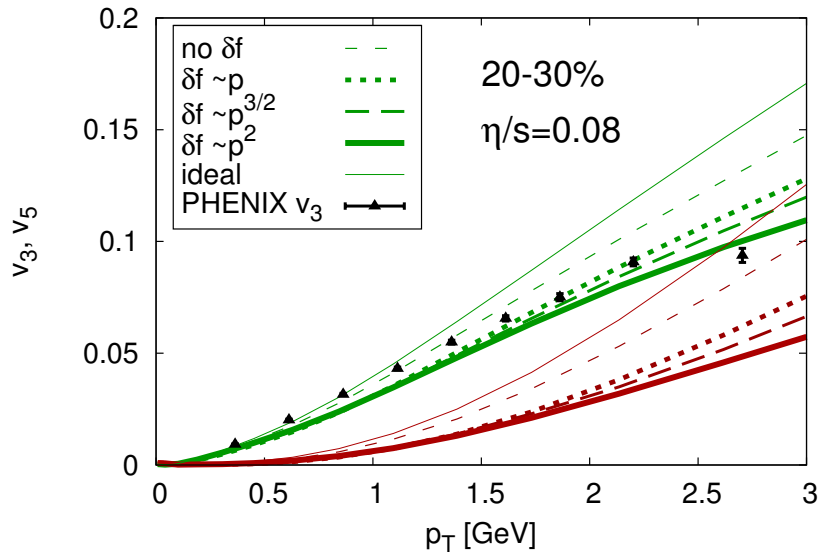
$$\delta f = f_0(1 \pm f_0) p^\alpha p^\beta \pi_{\alpha\beta} \frac{1}{2(\epsilon + \mathcal{P})T^2}$$

More generally: from using different energy dep. of relaxation time

$$\delta f = \frac{120}{\Gamma(6-\alpha)} f_0(1 \pm f_0) \left(\frac{T}{E}\right)^\alpha p^\alpha p^\beta \pi_{\alpha\beta} \frac{1}{2(\epsilon + \mathcal{P})T^2}$$

with $\alpha \in [0, 1]$.

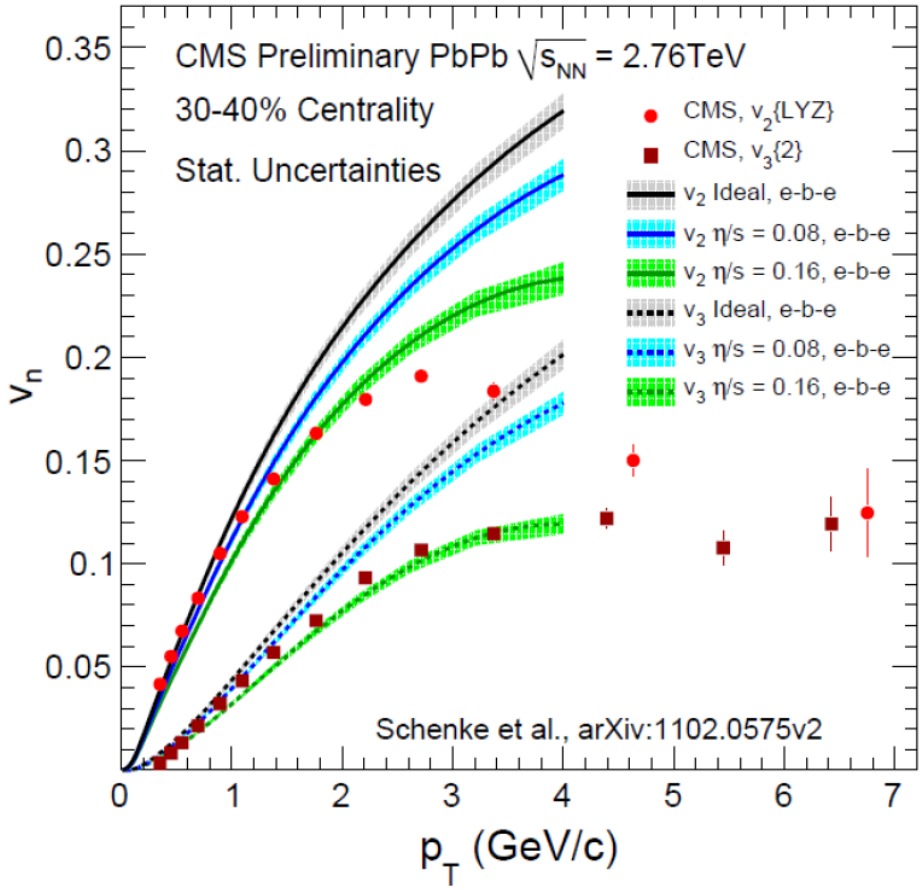




below $2\,\mathrm{GeV}$: weak dependence

LHC $v_2(p_T)$ and $v_3(p_T)$

v_3 was a prediction!

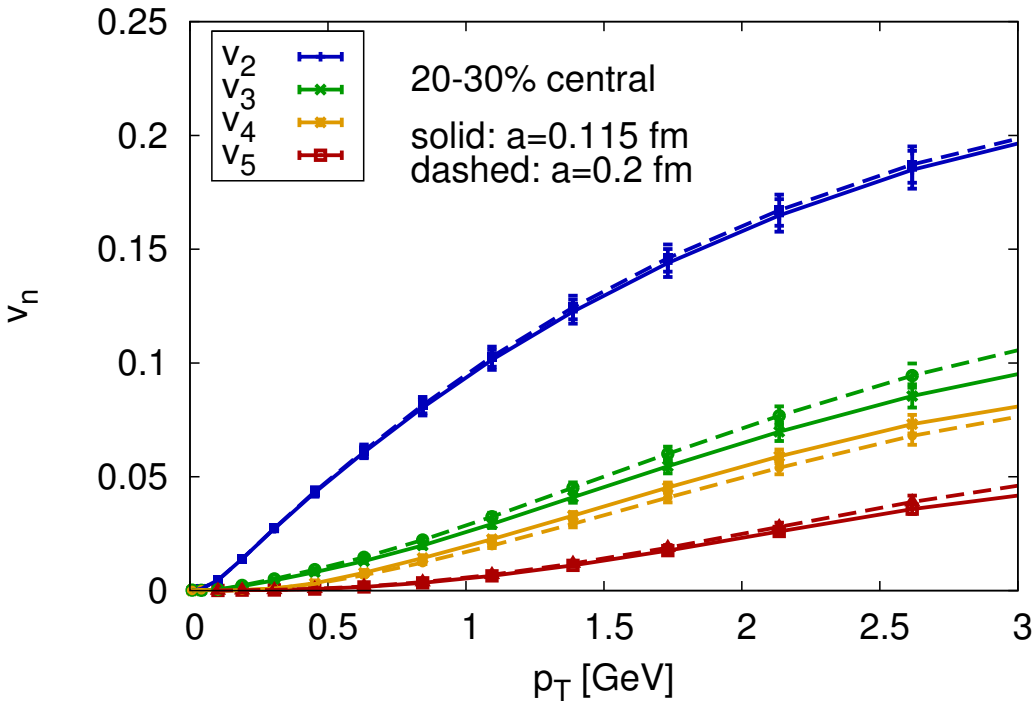


No perfect agreement. But not a lot of tuning.
This is $\langle v_n \rangle$, experiment is not.

Experimental data: The CMS collaboration 2011
B. Schenke, S. Jeon, C. Gale, Phys. Lett. B702, 59-63 (2011)

Sensitivity to the lattice spacing

Higher v_n are driven by fine spatial structures. Are we resolving these?

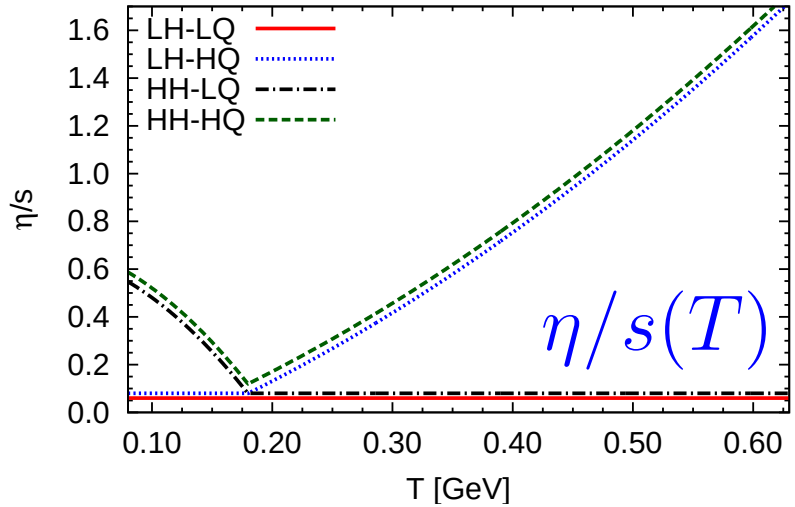


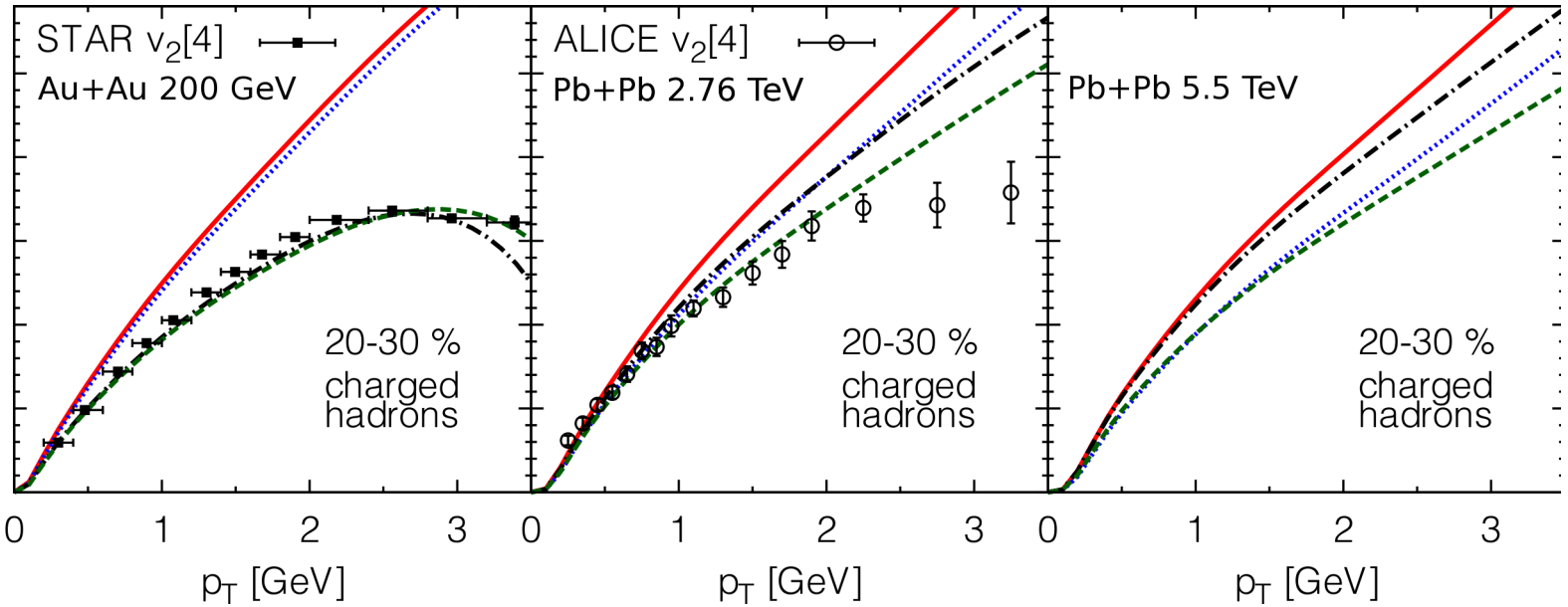
With finite viscosity (here $\eta/s = 0.08$) we are ok up to at least v_5!

Beyond constant η/s

Determine dependence of v_2 on modeled $\eta/s(T)$.
L=low, H="high"
H=hadronic phase, Q=QGP

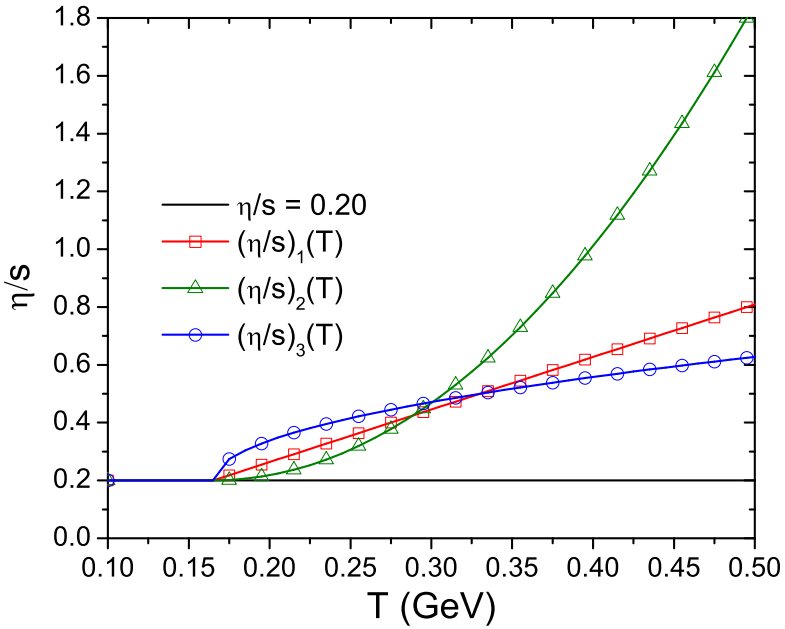
H. Niemi at al, Phys.Rev.Lett. 106 (2011) 212302

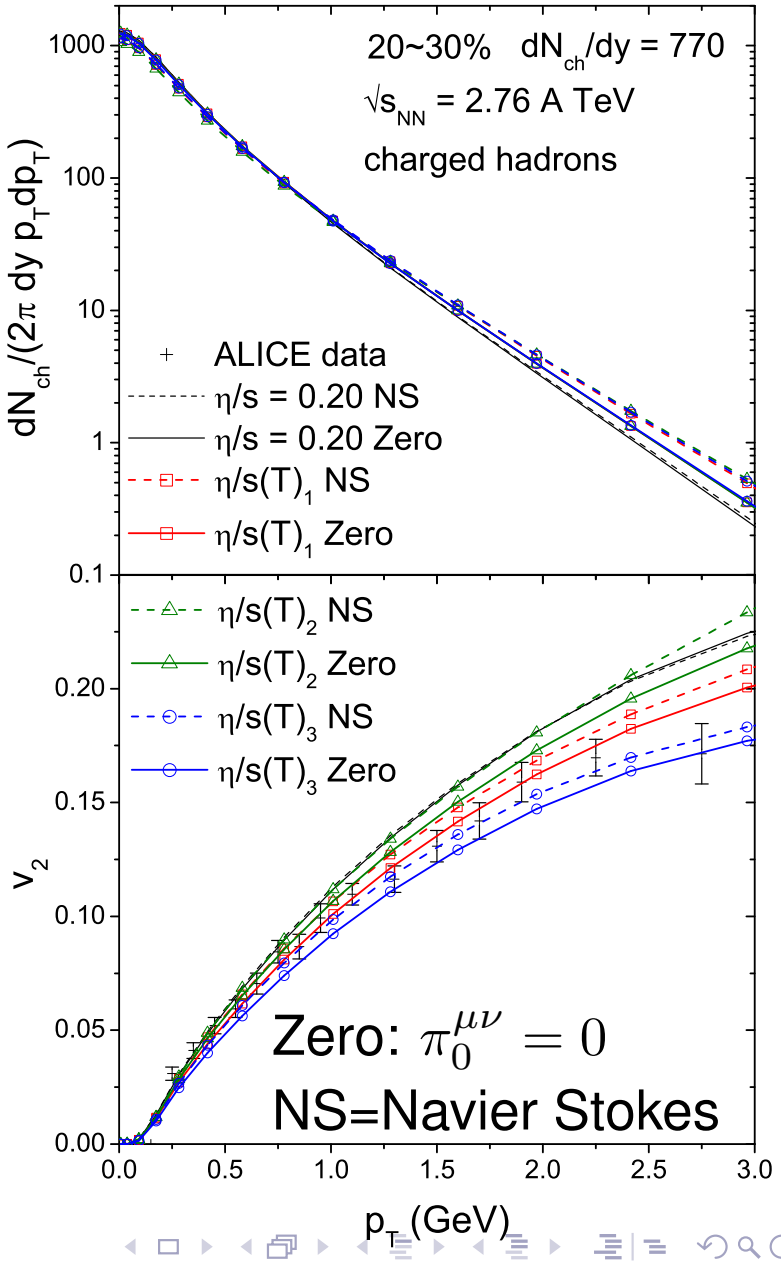




Weak dependence on QGP $\eta/s(T)$ at RHIC. Dependent on minimum.
Different at LHC energies (longer QGP phase, smaller gradients in the hadronic phase)

Beyond constant η/s





Strong dependence on the initial $\pi_0^{\mu\nu}$

by influencing the initial effective pressure profile.

Need better understanding of the pre-thermal evolution and its matching to viscous hydrodynamics.

C. Shen, U. Heinz, P. Huovinen, H. Song, Phys.Rev. C84 (2011) 044903

Navier Stokes: $\pi_0^{\mu\nu} = \eta_0(\nabla^\mu u_0^\nu + \nabla^\nu u_0^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u_0^\alpha)$

Summary

- 3+1d relativistic event-by-event viscous hydrodynamic simulations
- Event-by-event hydro allows to compute all flow harmonics.
- All harmonics depend differently on η/s and the initial state!
- Need to study higher harmonics and their fluctuations to determine initial conditions and η/s separately.
- At LHC higher flow harmonics are already well measured. Now we can constrain the theory using all of them!
- Current results hint at $\langle \eta/s \rangle_{\mathrm{QGP}} < 2/4\pi$ (bulk?)
- Future improvements need to be:
  More sophisticated initial conditions, bulk viscosity, temperature dependent η/s, hadronic afterburner (rescattering), ...
- Details of the evolution will affect thermal photon and dilepton production more than anything else.

BACKUP

Freeze out: convert energy density to particles

We compute the thermal particle spectra using the **Cooper-Frye formula**

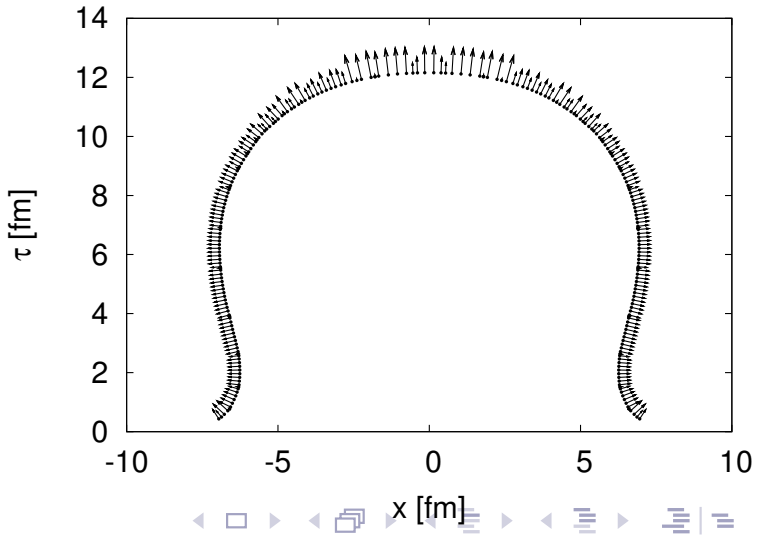
$$E\frac{dN}{dyp_T dp_T d\phi_p} = g_i \int_\Sigma f(u^\mu p_\mu) p^\alpha d^3\Sigma_\alpha \,,$$

with

$$f(u^\mu p_\mu) = \frac{1}{(2\pi)^3} \frac{1}{\exp\left[(u^\mu p_\mu - \mu_i)/T_{\rm FO}\right] \pm 1} \,,$$

and the freeze-out surface Σ.

Σ is defined as the surface of constant temperature $T_{\rm FO}$.
Below this temperature we assume free streaming of particles to the detector
+ resonance decays.



Viscosities of different fluids

Fluid	P (Pa)	T (K)	η (Pa s)	η/n ($\hbar$)	η/s ($\hbar/k_B$)
H_2O	0.1×10^6	370	2.9×10^{-4}	85	8.2
^4He	0.1×10^6	2.0	1.2×10^{-6}	0.5	1.9
H_2O	22.6×10^6	650	6.0×10^{-5}	32	2.0
^4He	0.22×10^6	5.1	1.7×10^{-6}	1.7	0.7
^6Li ($a = \infty$)	12×10^{-9}	23×10^{-6}	$\leqslant 1.7 \times 10^{-15}$	$\leqslant 1$	$\leqslant 0.5$
QGP	88×10^{33}	2×10^{12}	$\leqslant 5 \times 10^{11}$		$\leqslant 0.4$

New algorithm

We use a new (Kurganov-Tadmor) algorithm

- low numerical viscosity
- deals well with shocks

A. Kurganov, E. Tadmor, Journal of Computational Physics **160**, 241-282 (2000)

Finite volume method for Conservation Laws

How to solve the hyperbolic equations?

Example: 1-D current

- Average over a cell centered at x_j to turn

$$\partial_0 \rho = -\partial_x J$$

into

$$\frac{d}{dt}\bar{\rho}_j = -\frac{1}{\Delta x}\left(J(x_{j+1/2}) - J(x_{j-1/2})\right)$$

Exact so far. But J depends on $\rho(x_{j\pm 1/2})$, not $\bar{\rho}_j$.

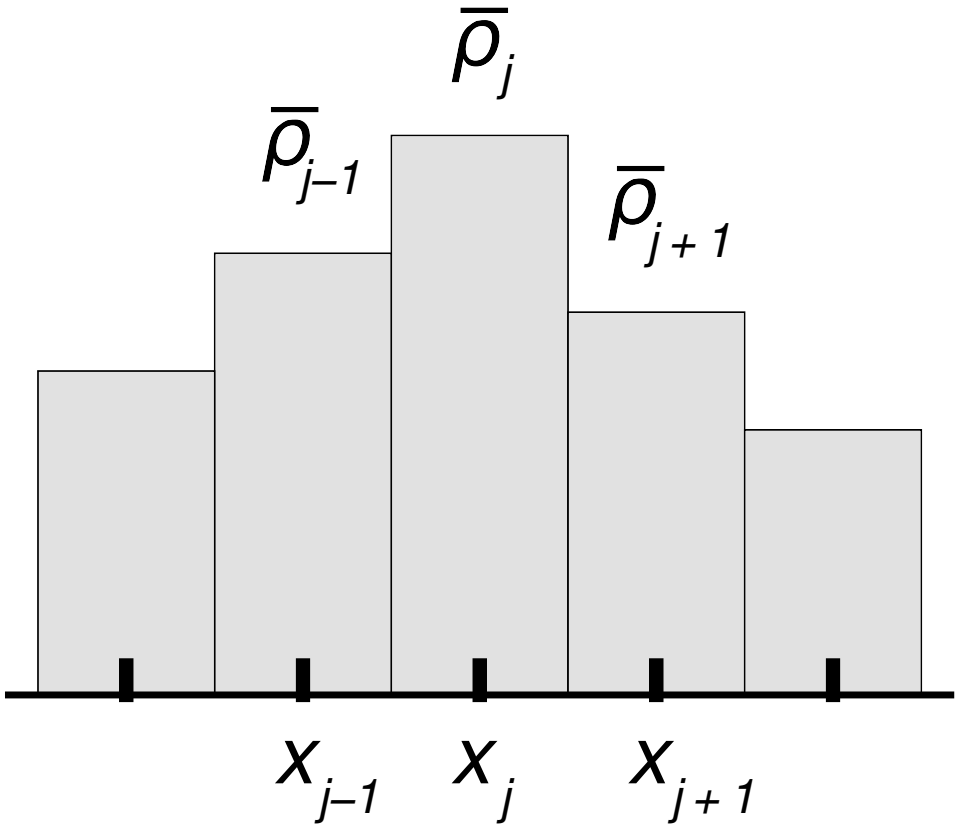
- Goal: Derive an approximate equation

$$\frac{d}{dt}\bar{\rho}_j = -\frac{1}{\Delta x}\left(H(x_{j+1/2}) - H(x_{j-1/2})\right)$$

where H is constructed using $\bar{\rho}_j$, not $\rho(x_{j\pm 1/2})$.

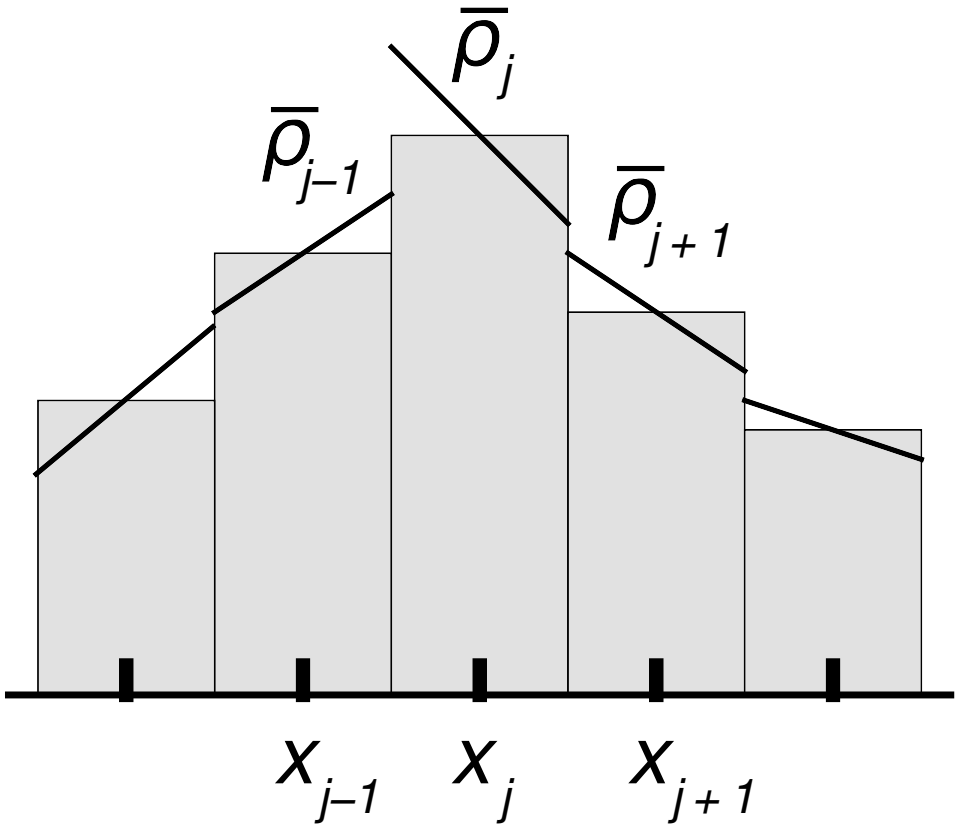
Finite volume method for Conservation Laws

- Main problem: How to get $J(x_{j\pm1/2})$ from $\bar{\rho}_j$?



Finite volume method for Conservation Laws

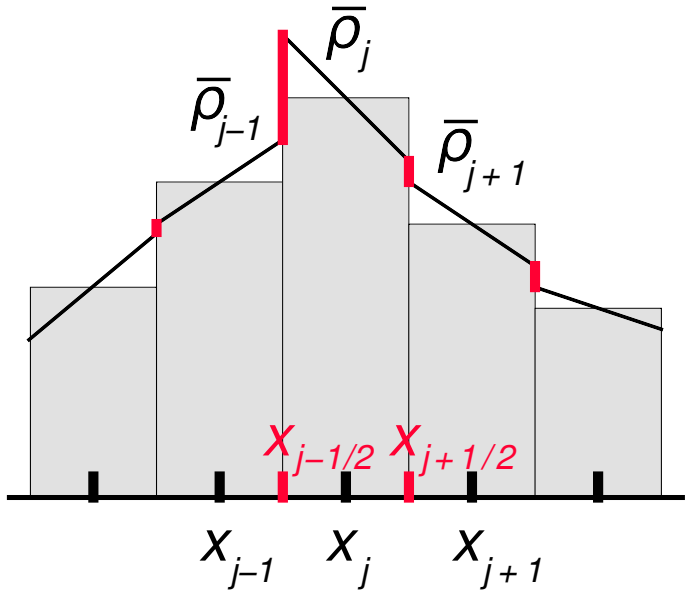
- Main problem: How to get $J(x_{j\pm1/2})$ from $\bar{\rho}_j$?



- Do a piecewise linear reconstruction of the local values.

Kurganov-Tadmor

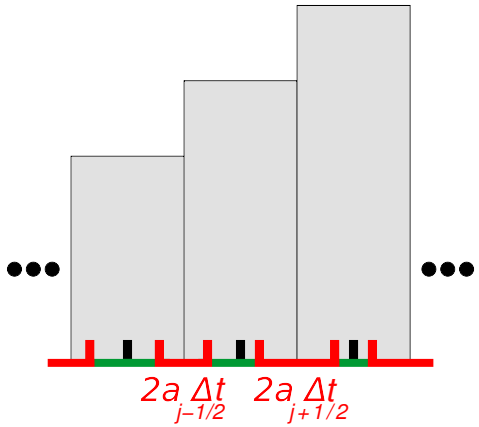
- **Trouble**: $x_{j\pm1/2}$ are where the boundaries are.
  Solve Riemann problems **or** avoid evaluating J at $x_{j\pm1/2}$



- **One idea**: Use staggered grid with cells centered at $x_{j-1/2}$ and $x_{j+1/2}$. So J is evaluated at x_j and x_{j+1}. In the next step use original grid, since discontinuities are now at x_j and x_{j+1}. Repeat (Nessyahu and Tadmor).
- **Better idea**: Use the propagation speed (Kurganov and Tadmor).

Kurganov-Tadmor

- Avoid evaluating J at discontinuity
- Divide intervals into $2a\Delta t$ around the discontinuity and the rest
- a: Maximum propagation speed



- Evaluate currents at edges of new cells
  (using charge conserving linear extrapolations)
- Project $a\Delta t$ intervals and $\Delta x - a\Delta t$ intervals
  onto the original Δx grid.

A. Kurganov, E. Tadmor, Journal of Computational Physics **160**, 241-282 (2000)

Kurganov-Tadmor

- In the end take $\Delta t \to 0$ limit to get

$$\frac{d}{dt}\bar{\rho}_j = -\frac{H_{j+1/2} - H_{j-1/2}}{\Delta x}$$

where

$$H_{j\pm1/2} = \frac{J(x_{j\pm1/2,+}) + J(x_{j\pm1/2,-})}{2} - \frac{a_{j\pm1/2}}{2}\left(\bar{\rho}_{j\pm1/2,+} - \bar{\rho}_{j\pm1/2,-}\right)$$

where

$$x^n_{j+1/2,\pm} = x_{j+1/2} \pm a^n_{j+1/2}\Delta t$$

and

$$\bar{\rho}_{j+1/2,+} = \bar{\rho}_{j+1} - \frac{\Delta x}{2}(\rho_x)_{j+1} \qquad \bar{\rho}_{j+1/2,-} = \bar{\rho}_j + \frac{\Delta x}{2}(\rho_x)_j$$

with the *minmod* flux limiter for the derivative $(\rho_x)_j$.

MUScl for Ion Collisions:

B. Schenke, S. Jeon, and C. Gale Phys. Rev. **C82**, 014903 (2010), arXiv:1004.1408

MUSCL = Monotonic Upstream Centered Scheme for Conservation Laws

- Solve KT

$$\frac{d}{dt}\bar{\rho}_j = -\frac{H_{j+1/2} - H_{j-1/2}}{\Delta x}$$

  with a second order Runge-Kutta scheme
- In 3+1 dimensions and $\tau - \eta$ coordinates
- Cooper-Frye freeze-out
  with sophisticated freeze-out surface construction
- Includes different equations of state including s95p-v1 from Huovinen and Petreczky (Lattice-QCD)

  P. Huovinen and P. Petreczky, Nucl. Phys. **A837**, 26-53 (2010)

Equations to be solved - ideal case

$$\partial_\mu T^{\mu\nu} = 0$$

In τ-η coordinates the relevant equations to be solved can be written as

$$\partial_\tau(\tau T^{\tau\tau}) + \partial_v(\tau T^{v\tau}) + \partial_{\eta_s}(T^{\eta_s\tau}) + T^{\eta_s\eta_s} = 0$$

and

$$\partial_\tau(\tau T^{\tau\eta_s}) + \partial_{\eta_s}(T^{\eta_s\eta_s}) + \partial_v(\tau T^{v\eta_s}) + T^{\tau\eta_s} = 0$$

and

$$\partial_\tau(\tau T^{\tau v}) + \partial_{\eta_s}(T^{\eta_s v}) + \partial_w(\tau T^{wv}) = 0$$

terms in red are expressed by the Kurganov-Tadmor term "$\frac{H_{j+1/2} - H_{j-1/2}}{\Delta x}$"

terms in green come from the coordinate transformation and are treated as external sources

Maximum propagation speed

Max speed in the k direction is the max eigenvalue of the Jacobian

$$\mathcal{J}^k_{ab} = \frac{\partial J^k_a}{\partial J^\tau_b}$$

a, b are five currents (net baryon, energy, and momentum).

Including viscosity

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu} + \pi^{\mu\nu}$$

$\pi^{\mu\nu}$ = shear part with $\pi^{\mu\nu} u_\nu = 0$ and $\pi^\mu_\mu = 0$.

We use the second order Israel-Stewart formalism, where $\pi^{\mu\nu}$ satisfies

$$\begin{aligned}\pi^{\mu\nu} =& \eta(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla_\alpha u^\alpha) \\ & - \frac{4}{3}\tau_\pi \pi^{\mu\nu}\partial_\alpha u^\alpha - \tau_\pi \Delta^\mu_\alpha \Delta^\nu_\beta u^\sigma \partial_\sigma \pi^{\alpha\beta} \,,\end{aligned}$$

with $\partial_\mu = \nabla_\mu + u_\mu(u_\alpha \partial^\alpha)$ and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$.

The relaxation time τ_π is set to $\frac{3\eta}{\epsilon + \mathcal{P}}$.

Equations to be solved - viscous case

In τ-η coordinates the relevant equations to be solved can be written explicitly as

$$\begin{aligned}&\partial_\tau(\tau T^{\tau\tau})+\partial_v(\tau T^{v\tau})+\partial_{\eta_s}(T^{\eta_s\tau})+T^{\eta_s\eta_s}\\&+\partial_\tau(\tau\pi^{\tau\tau})+\partial_v(\tau\pi^{v\tau})+\partial_{\eta_s}(\pi^{\eta_s\tau})+\pi^{\eta_s\eta_s}=0\end{aligned}$$

$$\begin{aligned}&\partial_\tau(\tau T^{\tau\eta_s})+\partial_v(\tau T^{v\eta_s})+\partial_{\eta_s}(T^{\eta_s\eta_s})+T^{\tau\eta_s}\\&+\partial_\tau(\tau\pi^{\tau\eta_s})+\partial_v(\tau\pi^{v\eta_s})+\partial_{\eta_s}(\pi^{\eta_s\eta_s})+\pi^{\tau\eta_s}=0\end{aligned}$$

$$\begin{aligned}&\partial_\tau(\tau T^{\tau v})+\partial_w(\tau T^{wv})+\partial_{\eta_s}(T^{\eta_s v})\\&+\partial_\tau(\tau\pi^{\tau v})+\partial_w(\tau\pi^{wv})+\partial_{\eta_s}(\pi^{\eta_s v})=0\end{aligned}$$

Kurganov-Tadmor current
treated as sources

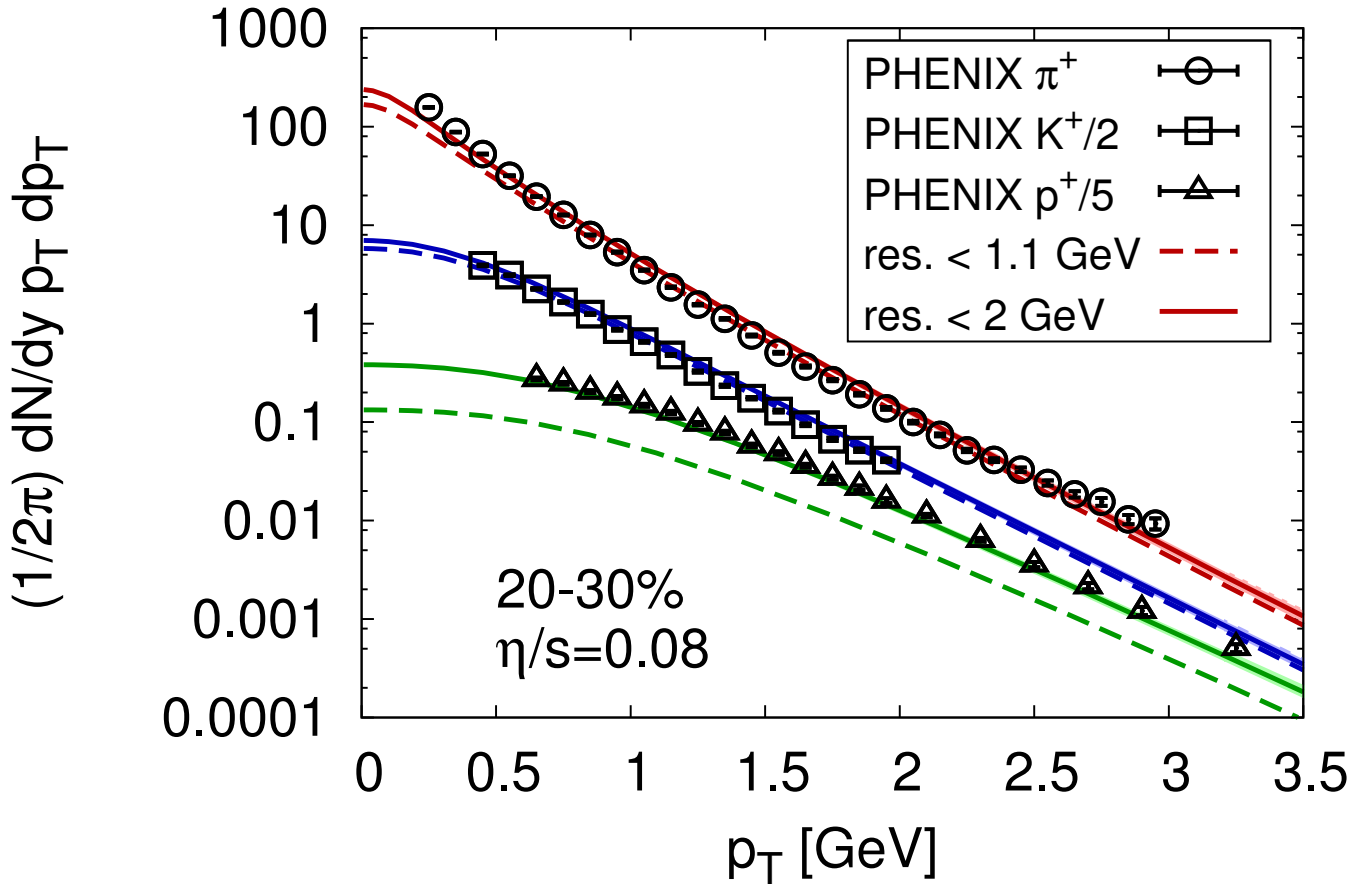
Equations to be solved - viscous case

In addition, we have to solve the equation for $\pi^{\mu\nu}$:

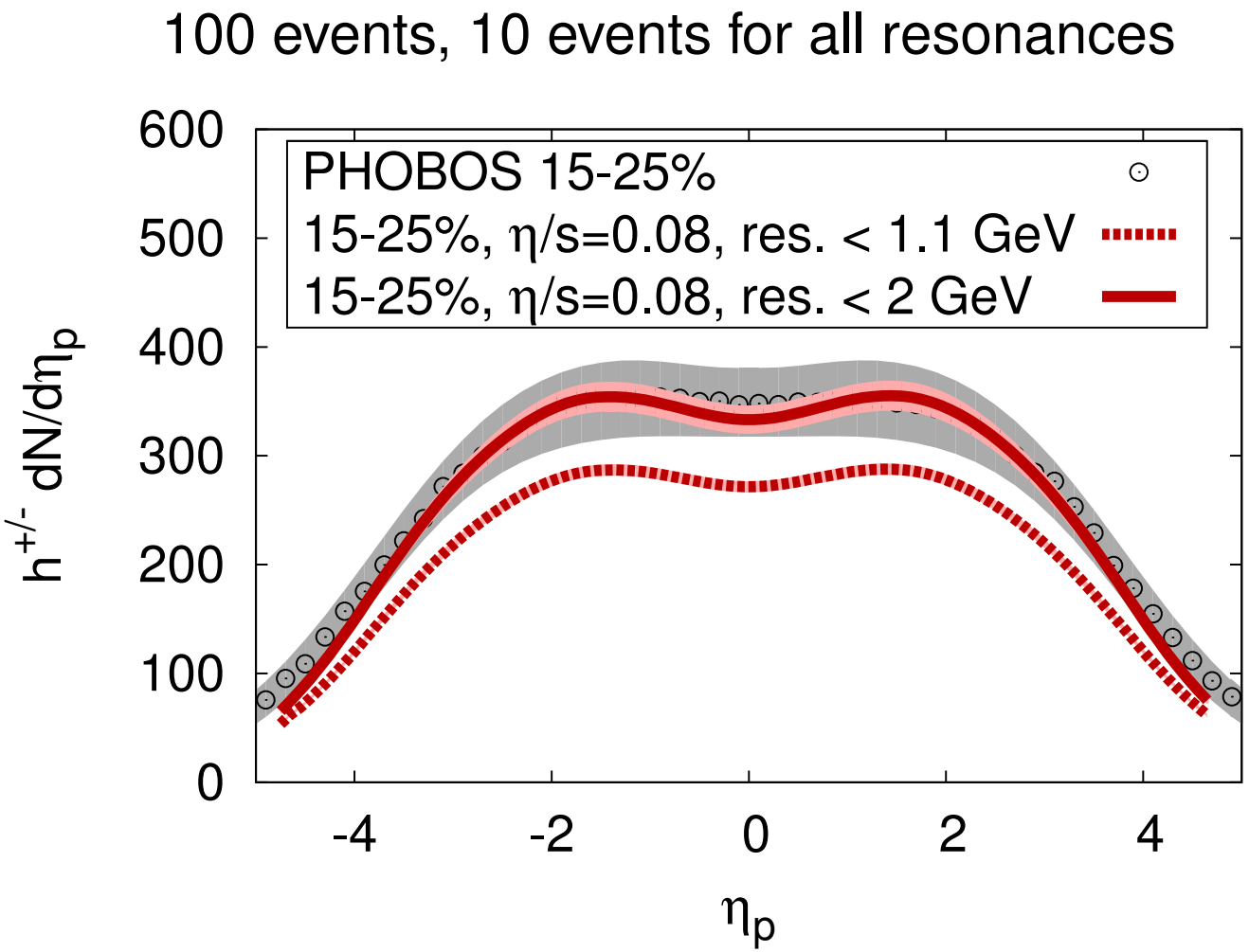
$$\begin{aligned}
\partial_c(u^c\pi^{ab}) = &-\frac{1}{2\tau}u^\tau\pi^{ab} + \frac{1}{\tau}\Delta^{a\eta}u^\eta\pi^{b\tau} - \frac{1}{\tau}\Delta^{a\tau}u^\eta\pi^{b\eta} \\
&- g_{cf}\pi^{cb}u^a Du^f - \frac{\pi^{ab}}{2\tau_\pi} - \frac{1}{6}\pi^{ab}\partial_c u^c \\
&+ \frac{\eta}{\tau_\pi}\left(-\frac{1}{\tau}\Delta^{a\eta}g^{b\eta}u^\tau + \frac{1}{\tau}\Delta^{a\eta}g^{b\tau}u^\tau\right. \\
&\qquad \left. + g^{ac}\partial_c u^b - u^a Du^b - \frac{1}{3}\Delta^{ab}\partial_c u^c\right) \\
&+ (a \leftrightarrow b)\,,
\end{aligned}$$

Spectra from e-b-e viscous MUSIC

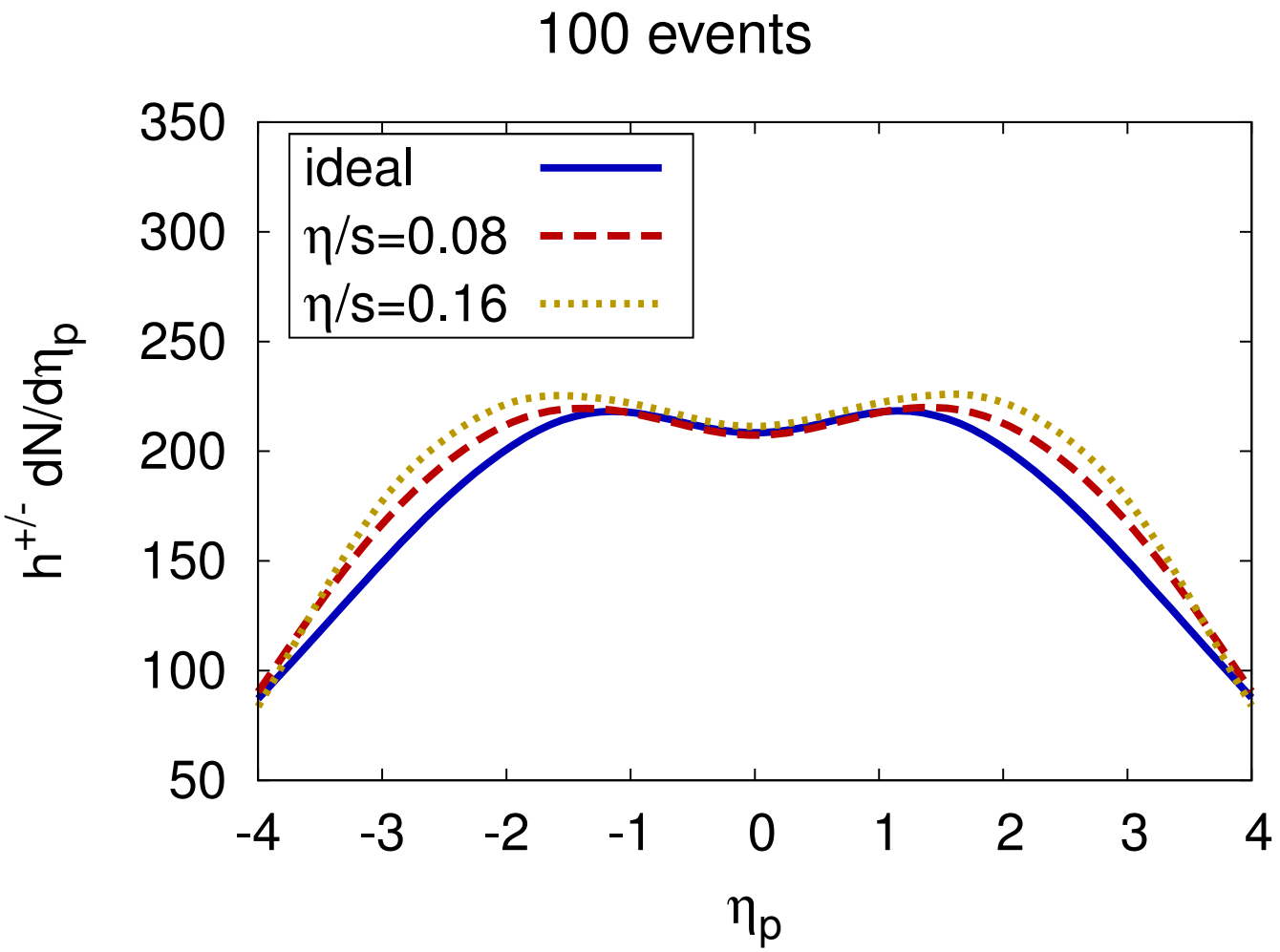
$\eta/s = 0.08$, 100 events, 10 events for all resonances



Spectra from e-b-e viscous MUSIC

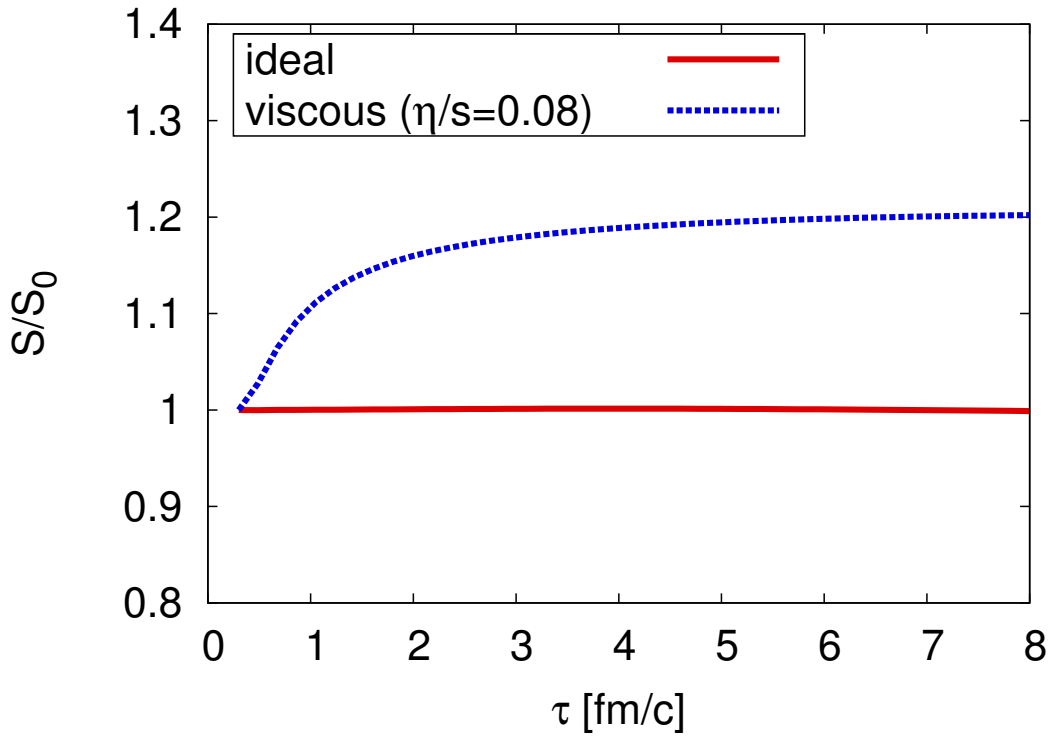


Effect of viscosity on longitudinal dynamics



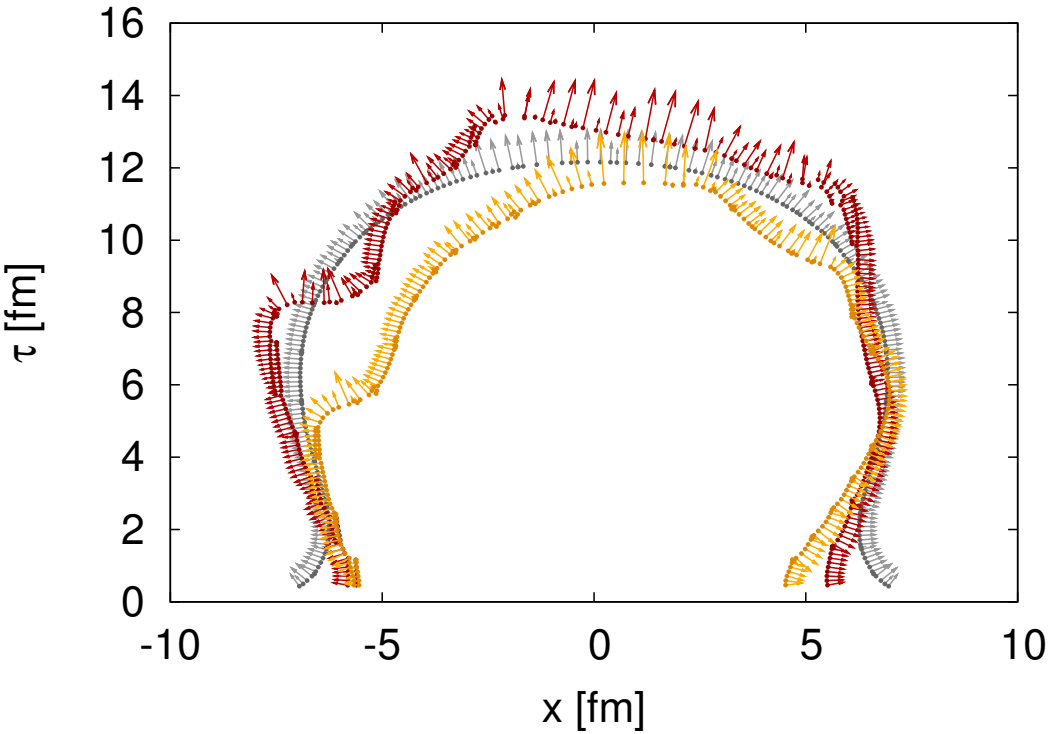
Entropy production

Ideal hydro: $\partial_\mu S^\mu = 0$ with $S^\mu = su^\mu \Rightarrow S = \int_V d\mathbf{x}_\perp \tau d\eta s u^\tau = \text{const.}$
Viscous hydro: physical entropy production



Freeze-out surface in a single event

The freeze-out finder works great in a single event:



B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

Issues

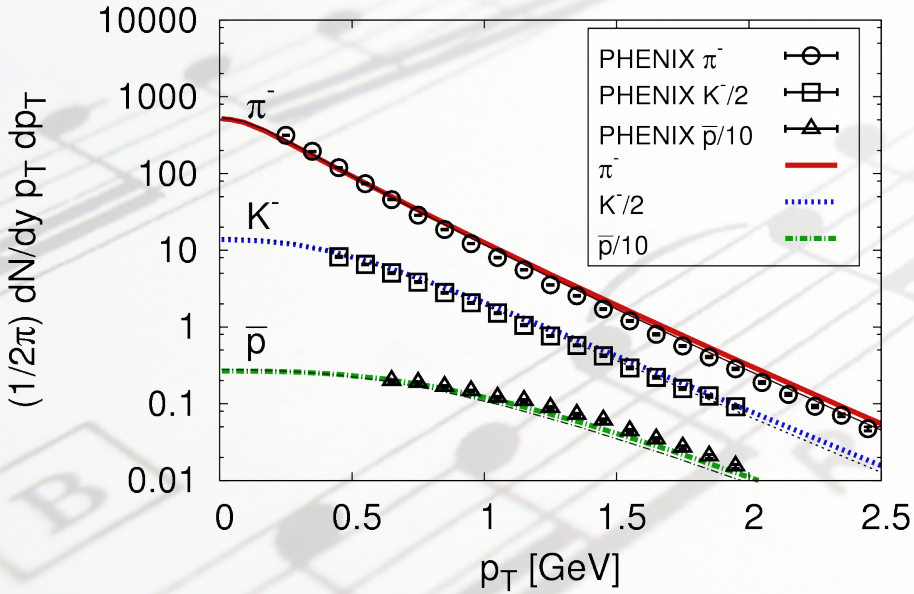
What if $\pi^{\mu\nu}$, the viscous correction to $T^{\mu\nu}$, becomes larger than $T^{\mu\nu}$?
It does happen. Mostly at edges where the energy density is low.

Options:

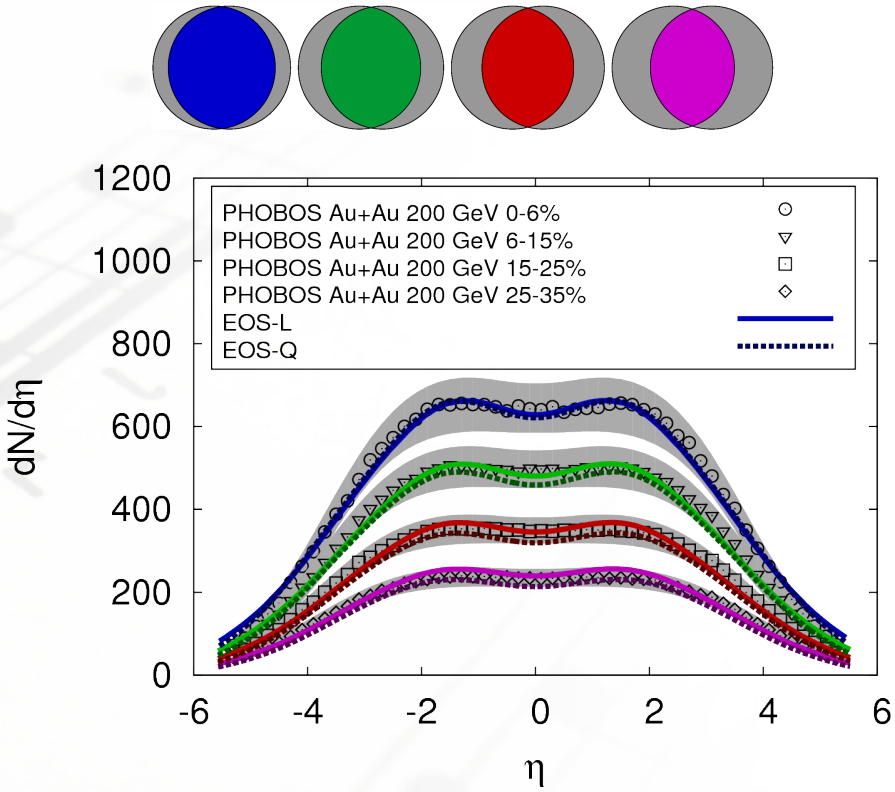
- Revert to previous value of $\pi^{\mu\nu}$:
  Will stop evolution of $\pi^{\mu\nu} \rightarrow$ Overestimates $\pi^{\mu\nu}$ at freeze-out.
- Set $\pi^{\mu\nu}$ to zero:
  Numerically unstable. Underestimates viscous contribution.
- Return to previous value and reduce it by 5%:
  Best tested method. Smoothly decreases $\pi^{\mu\nu}$. Numerically stable.

Results from ideal MUSIC

Include resonances up to $2\,\mathrm{GeV}$ and compute their decays to get the Particle spectra



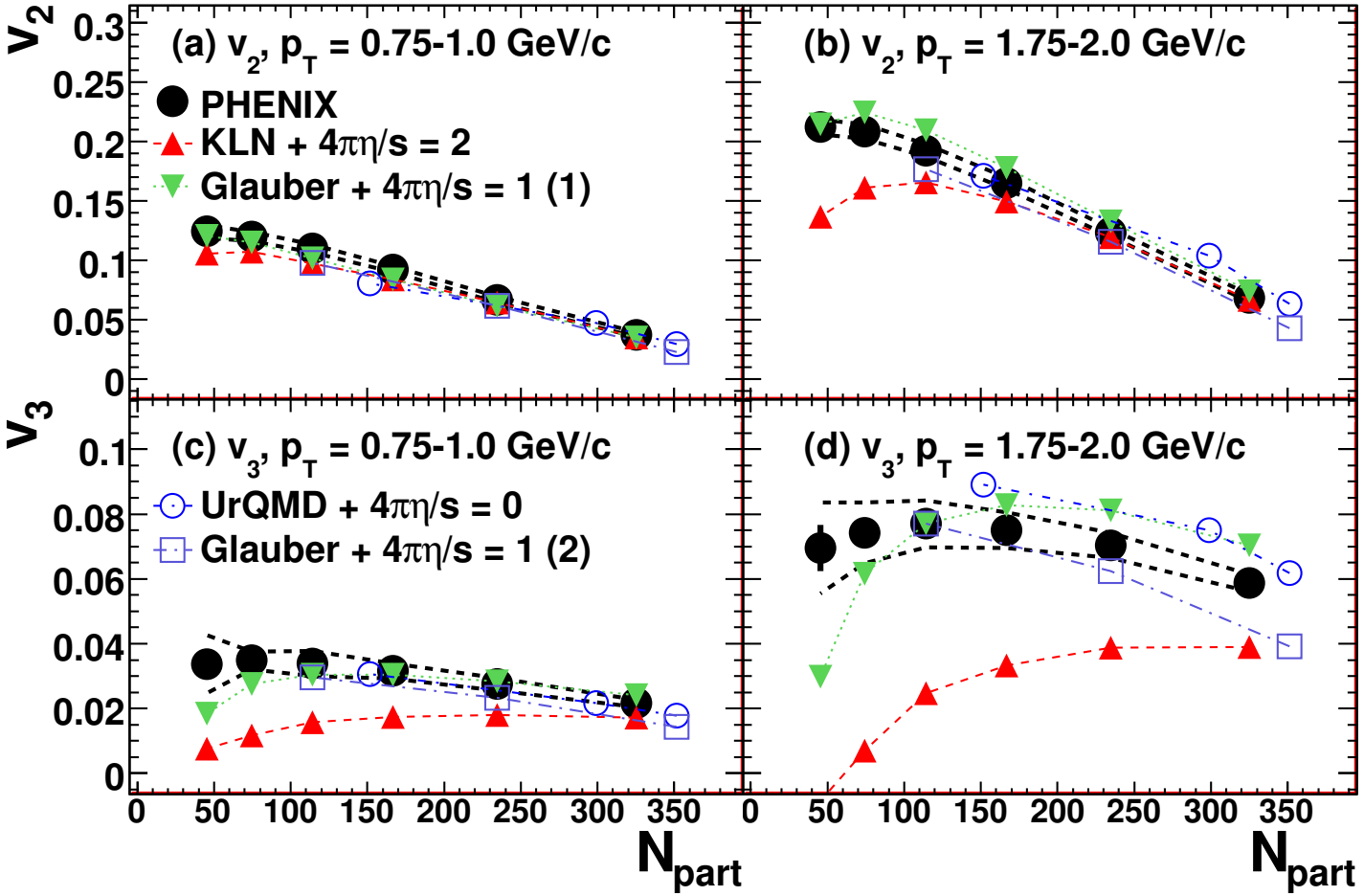
transverse momentum



pseudo-rapidity

B. Schenke, S. Jeon, and C. Gale Phys. Rev. **C82**, 014903 (2010)

Flow results from e-b-e viscous MUSIC

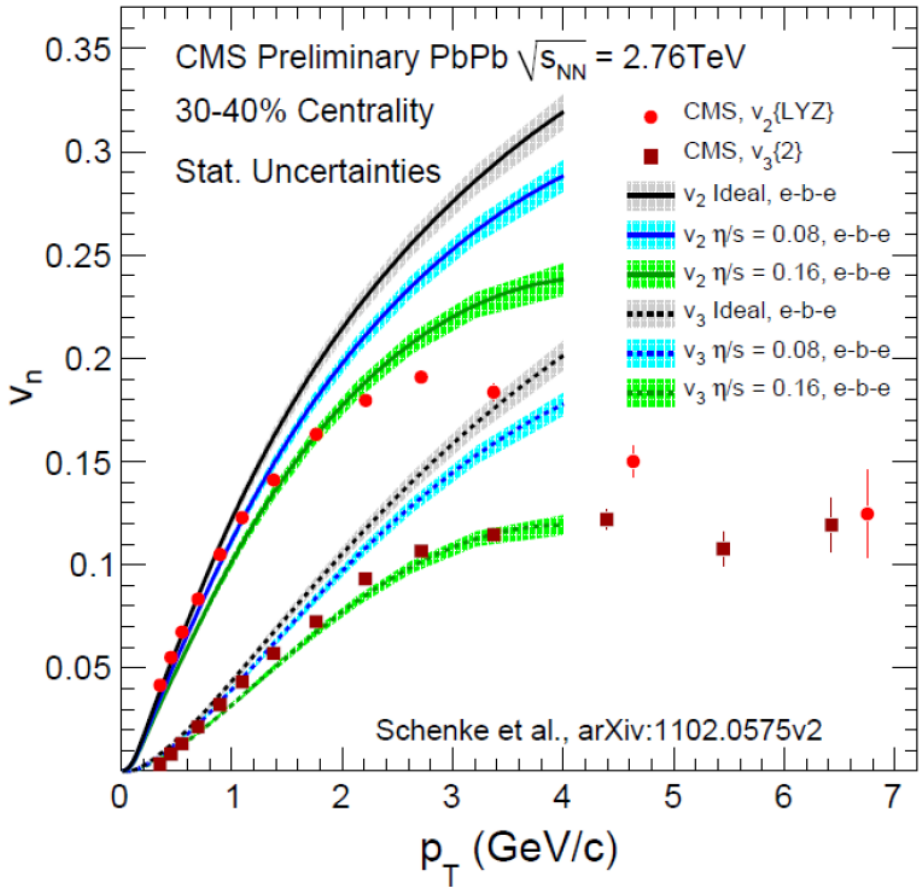


blue squares are MUSIC

B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)
Experimental data: A. Adare et al. (PHENIX), arXiv:1105.3928 (2011)

LHC $v_2(p_T)$ and $v_3(p_T)$

v_3 was a prediction!



No perfect agreement yet. But not a lot of tuning was done for LHC.

Experimental data: The CMS collaboration 2011

B. Schenke, S. Jeon, C. Gale, Phys. Lett. B702, 59-63 (2011)

Numerical viscosity

- In the Lax scheme, the time derivative is expressed by

$$\frac{d}{dt}\rho = \frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} - \frac{\rho_{j+1}^n - 2\rho_j^n + \rho_{j-1}^n}{2\Delta t}$$

  to deal with instability of the simplest discretization (FTCS).
  In "continuum form" the last term reads $\frac{(\Delta x)^2}{2\Delta t}\partial_x^2\rho$, introducing numerical viscosity
- In KT we have

$$\frac{d}{dt}\bar{\rho}_j(t) = -\frac{\left[J\left(\bar{\rho}^+_{j+1/2}(t)\right) + J\left(\bar{\rho}^-_{j+1/2}(t)\right)\right] - \left[J\left(\bar{\rho}^+_{j-1/2}(t)\right) + J\left(\bar{\rho}^-_{j-1/2}(t)\right)\right]}{2\Delta x}$$
$$+\frac{1}{2\Delta x}\left\{a_{j+1/2}(t)\left[\bar{\rho}^+_{j+1/2}(t) - \bar{\rho}^-_{j+1/2}(t)\right] - a_{j-1/2}(t)\left[\bar{\rho}^+_{j-1/2}(t) - \bar{\rho}^-_{j-1/2}(t)\right]\right\}$$

  Taylor expanding $\bar{\rho}_{j\pm1}$ in $\bar{\rho}^{\pm}_{j\pm1/2}$ around $\bar{\rho}_j$ shows that numerical viscosity is of order $(\Delta x)^3$
  No $1/\Delta t$ term!

Maximum propagation speed

Max speed in the k direction is the max eigenvalue of the Jacobian

$$\mathcal{J}^k_{ab} = \frac{\partial J^k_a}{\partial J^\tau_b}$$

a, b are five currents (net baryon, energy, and momentum).
For $k = x, y$:

- Two eigenvalues are u^k/u^τ.
- Two eigenvalues are $\lambda^\pm_k = \frac{A \pm \sqrt{B}}{D}$,

  with $A = u^\tau u^k (1 - v_s^2)$, $B = [u_\tau^2 - u_k^2 - (u_\tau^2 - u_k^2 - 1)v_s^2]v_s^2$, $D = u_\tau^2 - (u_\tau^2 - 1)v_s^2$

For $k = \eta_s$:

- Same eigenvalues but scaled by $1/\tau$

Largest eigenvalue is $\lambda^\pm_k = \frac{|A| + \sqrt{B}}{D}$ (times $1/\tau$ if $k = \eta_s$).
That's the maximum local propagation speed.

Minmod flux limiter

The minmod flux limiter takes care of switching between a second order derivative where the function is smooth and a first order derivative where there are large gradients.

The derivative is

$$(\rho_x)_j = \text{minmod}\left(\theta\frac{\bar{\rho}_{j+1}-\bar{\rho}_j}{\Delta x}, \frac{\bar{\rho}_{j+1}-\bar{\rho}_{j-1}}{2\Delta x}, \theta\frac{\bar{\rho}_j-\bar{\rho}_{j-1}}{\Delta x}\right),$$

with

$$\text{minmod}(x_1, x_2, \cdots) = \begin{cases} \min_i\{x_i\}, & \text{if } x_i > 0\ \forall i \\ \max_i\{x_i\}, & \text{if } x_i < 0\ \forall i \\ 0, & \text{otherwise} \end{cases}$$

$\theta \in [1, 2]$. We use $\theta = 1.1$.

Consistent gradient expansion: second order

The most general expression to second order in gradients in curved space for a conformal theory is

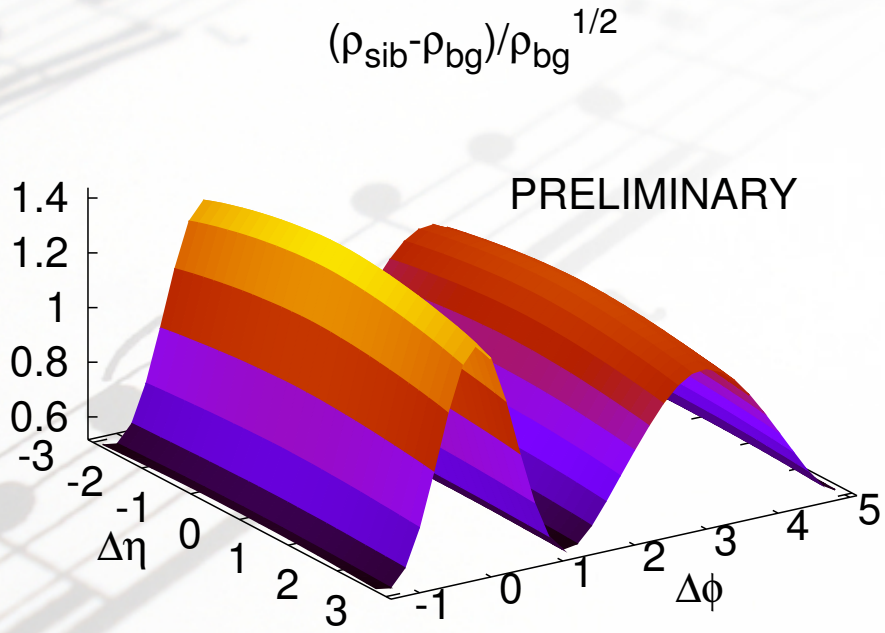
$$\pi^{\mu\nu} = \eta\nabla^{<\mu}u^{\nu>} - \tau_\pi\left[\Delta^\mu_\alpha\Delta^\nu_\beta D\pi^{\alpha\beta} + \frac{4}{3}\pi^{\mu\nu}(\nabla_\alpha u^\alpha)\right]$$
$$+\frac{\kappa}{2}\left[R^{<\mu\nu>} + 2u_\alpha R^{\alpha<\mu\nu>\beta}u_\beta\right]$$
$$-\frac{\lambda_1}{2\eta^2}\pi^{<\mu}{}_\lambda\pi^{\nu>\lambda} - \frac{\lambda_2}{2\eta}\pi^{<\mu}{}_\lambda\Omega^{\nu>\lambda} - \frac{\lambda_3}{2}\Omega^{<\mu}_\lambda\Omega^{\nu>\lambda}$$

where $R^{\alpha\beta\gamma\delta}$ is the Riemann tensor, $R^{\mu\nu}$ the Ricci tensor, and τ_π, κ, $\lambda_1, \lambda_2, \lambda_3$ are five independent second order transport coefficients.

Baier, Romatschke, Son, Starinets, Stephanov, JHEP 0804, 100 (2008)

Correlations: away-side ridge

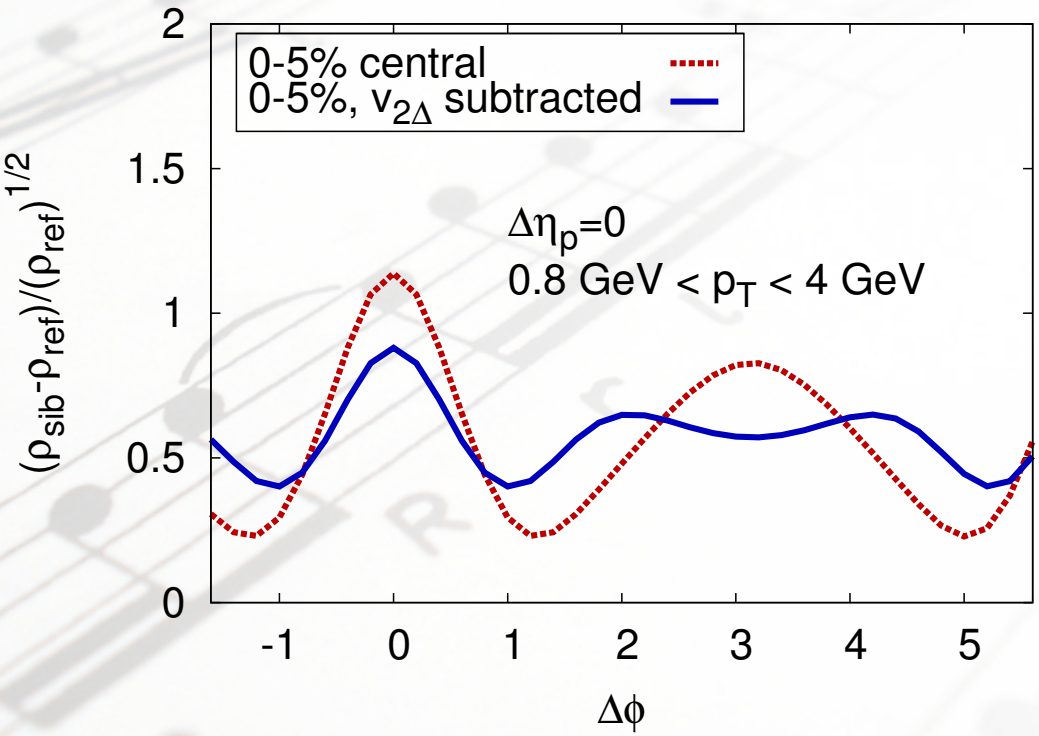
Untriggered $\Delta\eta - \Delta\phi$ correlations from viscous hydro.



Correlations: Fourier decomposition

Fourier decomposition, subtract the elliptic flow component:

as in B. Alver and G. Roland, Phys.Rev.C81:054905 (2010)



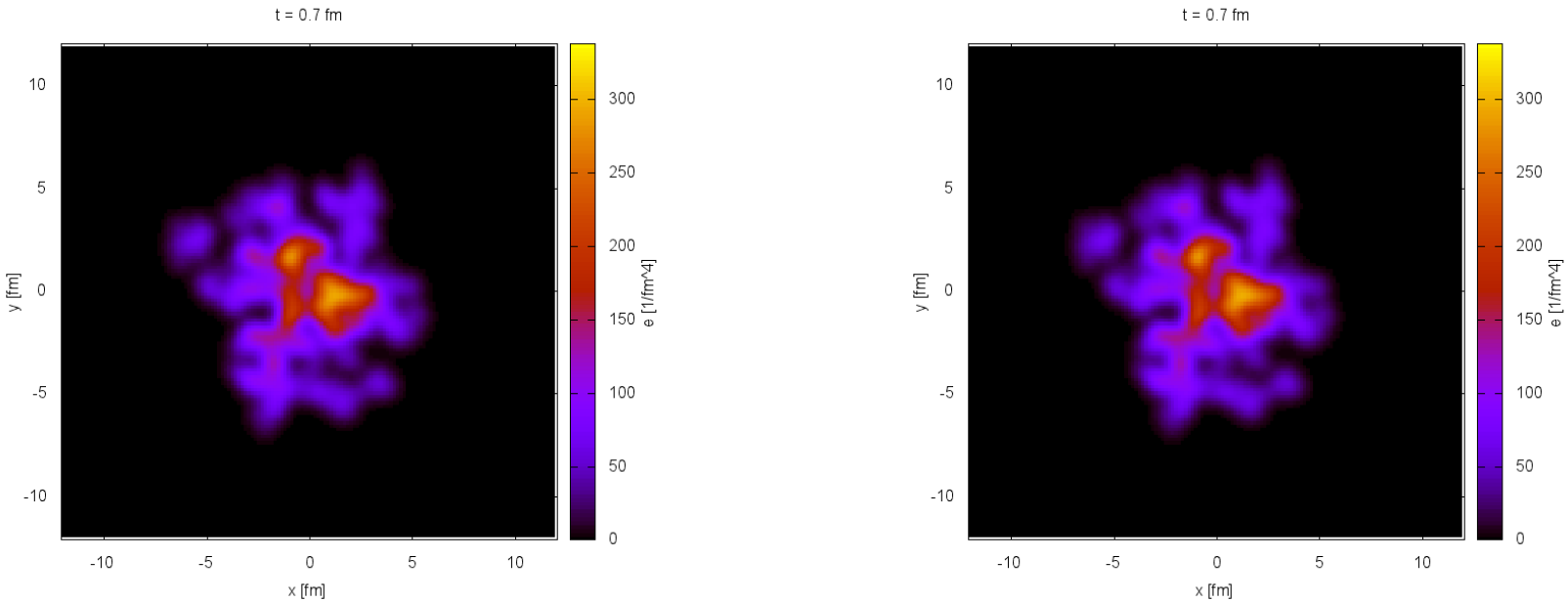
PRELIMINARY

Viscosity in event-by-event simulations

(scale in 1/fm^4):

ideal

$\eta/s = 0.16$

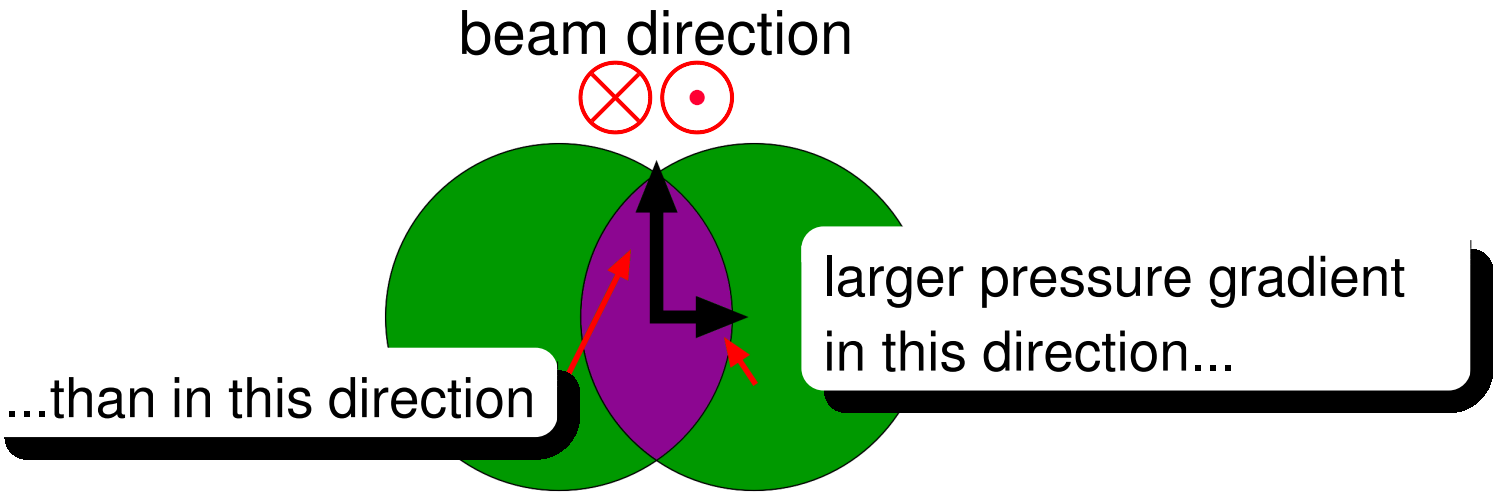


(note that the scale is dynamically adjusted)

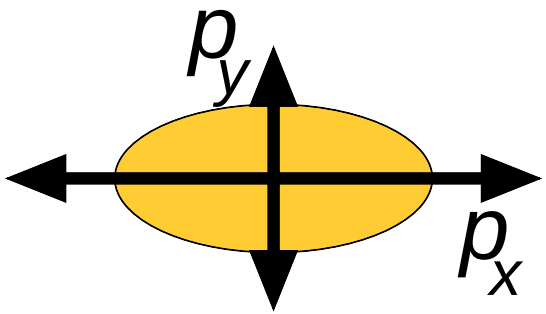
B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

Anisotropic flow: Remember elliptic flow

Non-central collision



Particle distribution in **momentum space** will be anisotropic.
Quantify using a Fourier decomposition:

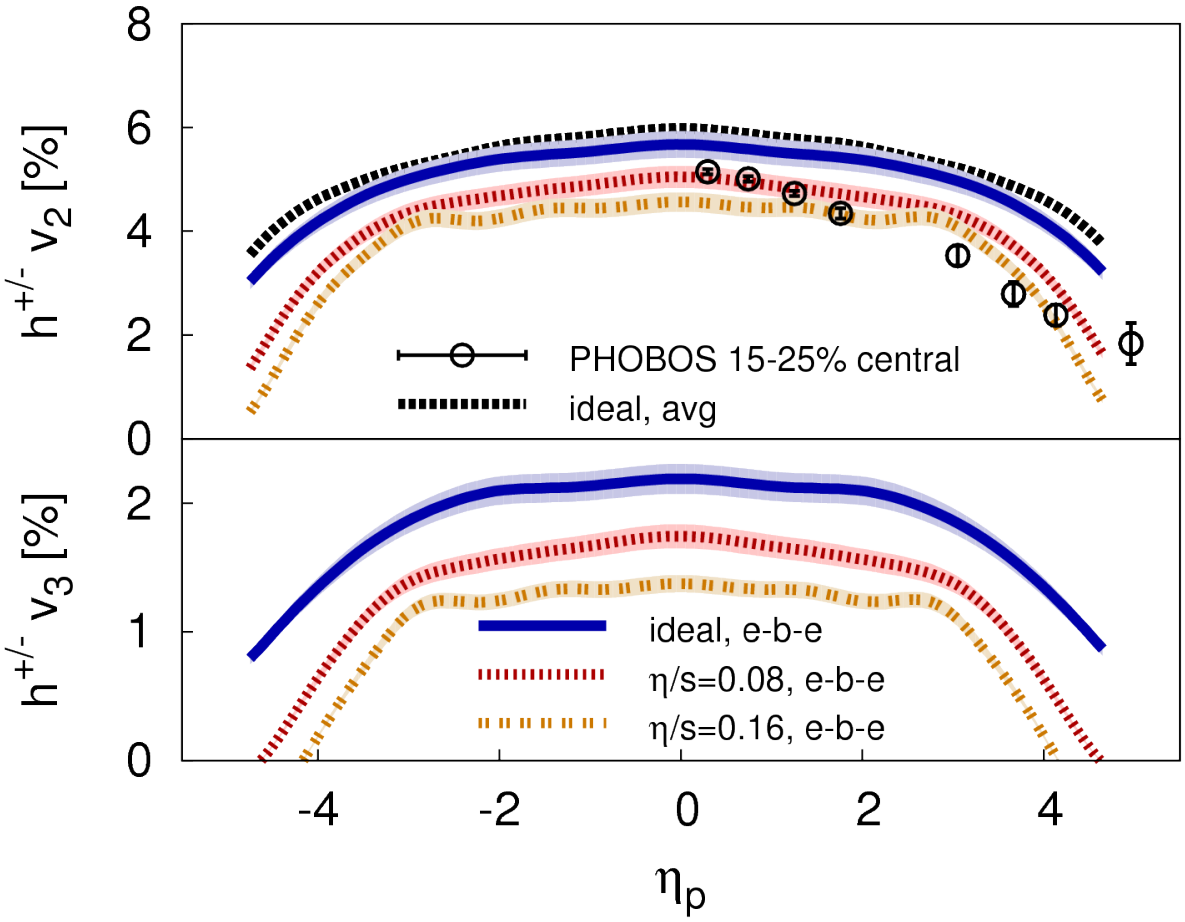


$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left(1 + \sum_n (2v_n \cos(n\phi))\right) \Rightarrow v_2 \text{ characterizes elliptic flow}$$

Flow results from e-b-e viscous MUSIC

Event-by-event fluctuations and viscosity improve agreement with experimental data.

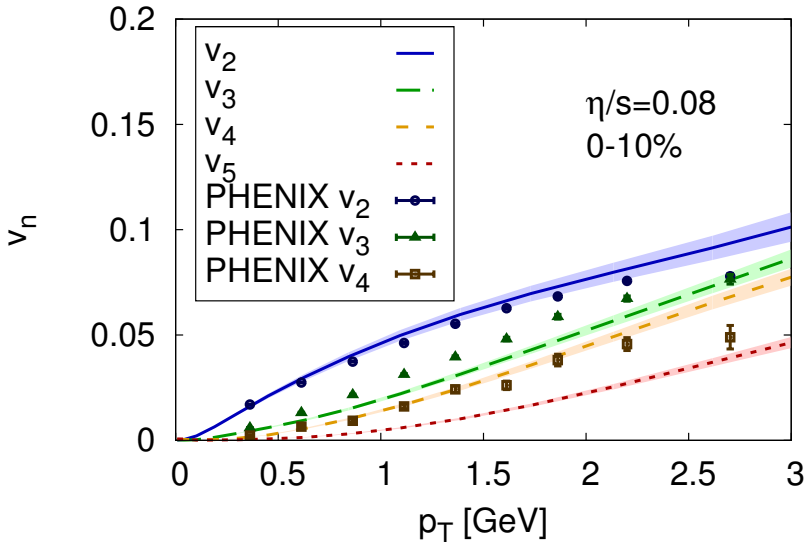
v_2 and v_3 as functions of pseudo-rapidity η_p

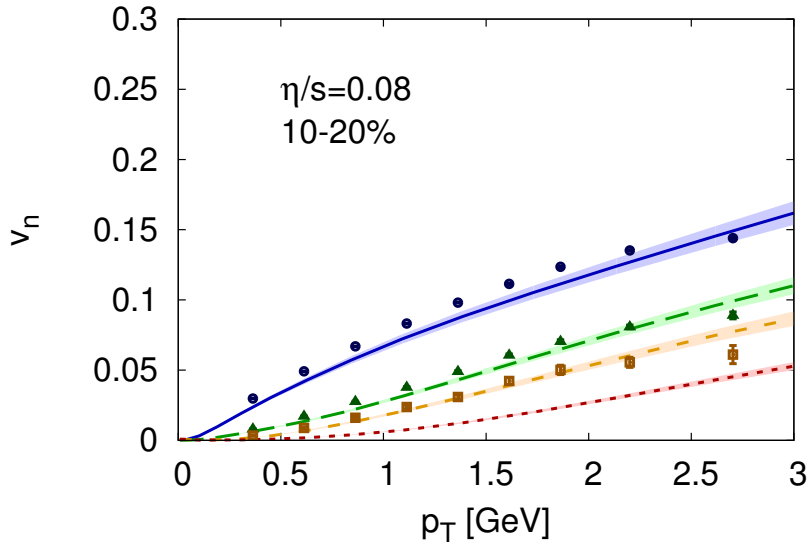


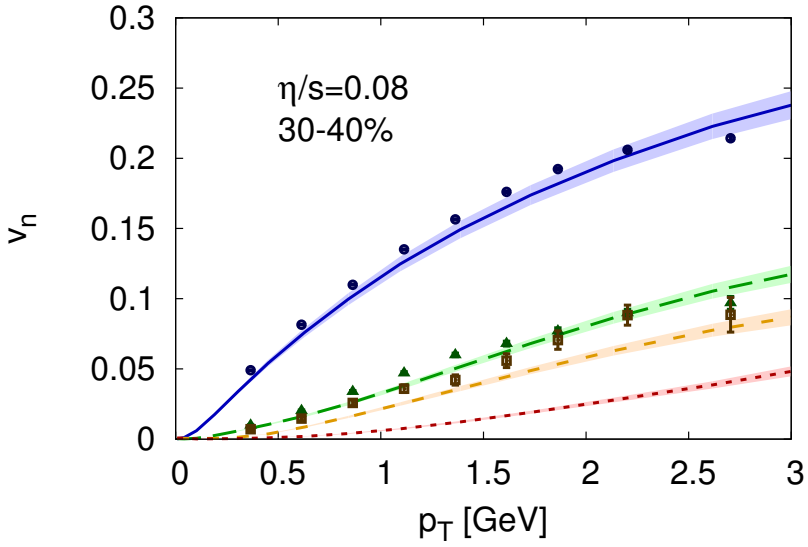
B. Schenke, S. Jeon, and C. Gale, PRL 106, 042301 (2011)

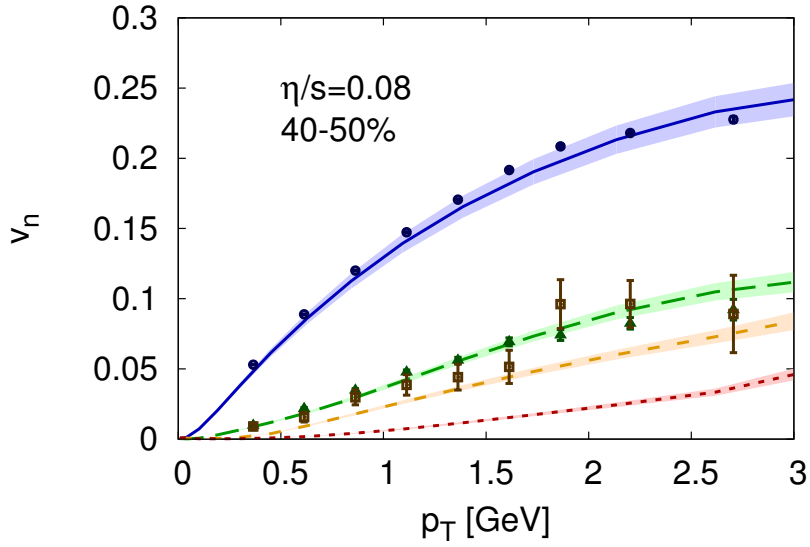
Experimental data: B.B. Back et al. (PHOBOS), Phys.Rev.C72:051901 (2005)

Centrality dependence

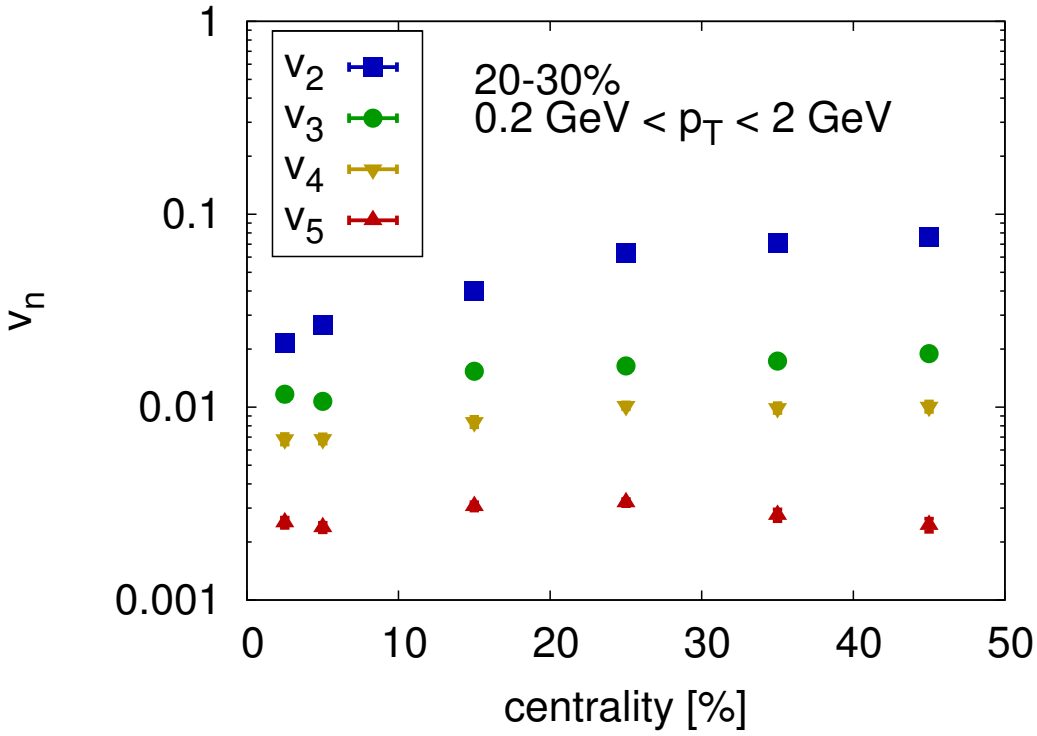








Centrality dependence



using $\eta/s = 0.08$.

Effect of δf

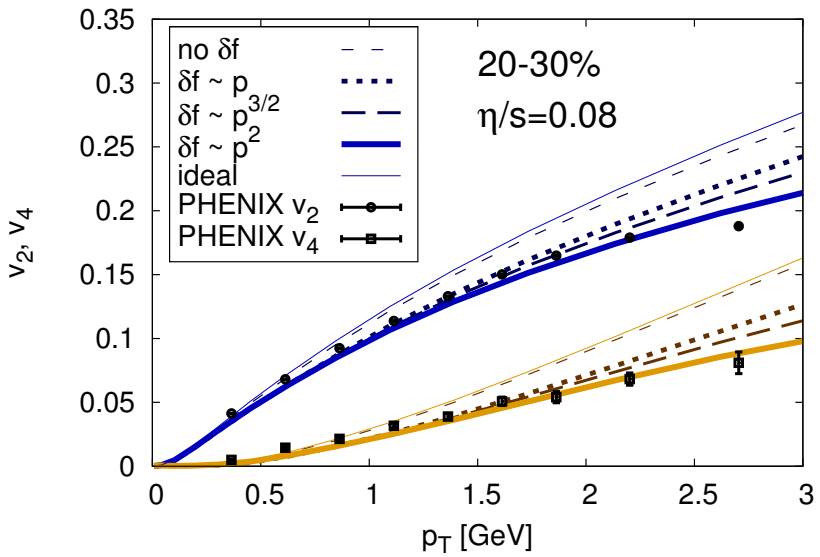
Remember: $$\delta f = f_0(1 \pm f_0) p^\alpha p^\beta \pi_{\alpha\beta} \frac{1}{2(\epsilon + \mathcal{P})T^2}$$

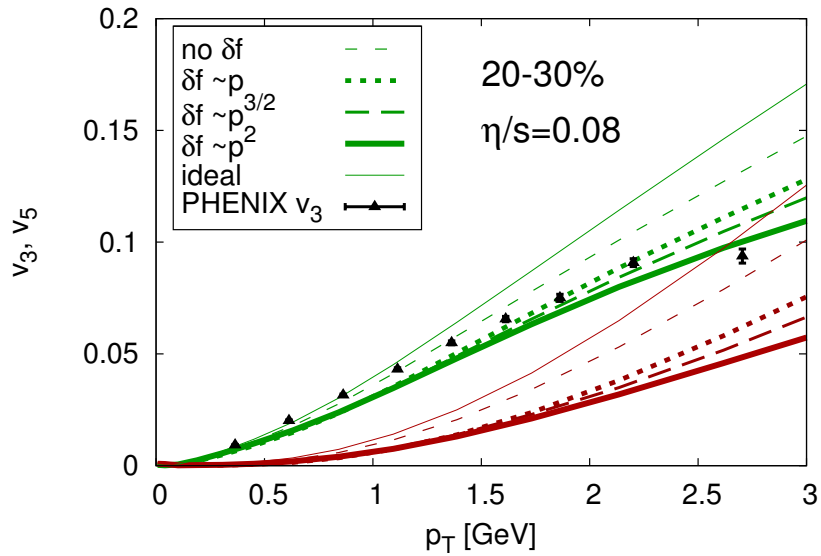
The choice $\delta f \sim p^2$ is not unique.

More generally: from using different energy dep. of relaxation time

$$\delta f = \frac{120}{\Gamma(6-\alpha)} f_0(1 \pm f_0) \left(\frac{T}{E}\right)^\alpha p^\alpha p^\beta \pi_{\alpha\beta} \frac{1}{2(\epsilon + \mathcal{P})T^2}$$

with $\alpha \in [0, 1]$.





below $2\,\mathrm{GeV}$: weak dependence