Estimating thermal dilepton rates and electrical conductivity in Quenched QCD

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Based on work with
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• **Introduction & Motivation**
  - thermal dilepton & photon emission rate, electrical conductivity
  - Euclidean correlation and spectral functions

• **Vector correlation function on the lattices**
  - finite volume & cut-off effects
  - thermal moments of spectral functions
  - continuum extrapolation

• **Thermal dilepton rates and electric conductivity**
  - chi-square fitting with Ansätz
  - Maximum Entropy Method analysis without Ansätz

• **Conclusions**
Dilepton rates

PHENIX: PRC 81(2010)034911

- pp data well described by cocktails
- enhancement in the low mass region in AuAu data
Dilepton rates

PHENIX: PRC 81(2010)034911

- Low mass region hadronic contribution not understood
- First principle calculation needed

\[ \frac{dN_{l^+l^-}}{d\omega d^3p} = C_{em} \frac{\alpha_{em}^2}{6\pi^3} \frac{\rho_V(\omega, \vec{p}, T)}{(\omega^2 - \vec{p}^2)(e^{\omega/T} - 1)} \]

\[ C_{em} = e^2 \sum_{f=1}^{n_f} Q_f^2 \]
- pp data well described by NLO pQCD at $p_T > 2$ GeV/c
- Enhanced direct photon production in AuAu at $p_T < 2.5$ GeV/c
Direct photon production

- pp data well described by NLO pQCD at $pt > 2\text{ GeV/c}$
- Enhanced direct photon production in AuAu at $pt < 2.5 \text{ GeV/c}$
- Non-perturbative calculation of photon emission from QCP needed

$$\omega \frac{dR_\gamma}{d^3p} = C_{em} \frac{\alpha_{em}}{4\pi^2} \frac{\rho_V(\omega = |\vec{p}|, T)}{e^{\omega/T} - 1}$$
Electrical conductivity

Electrical conductivity:

\[
\frac{\sigma}{T} = \frac{C_{em}}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega)}{\omega T}
\]

The emission rate of soft photons:

\[
\lim_{\omega \to 0} \omega \frac{dR_\gamma}{d^3p} = \frac{3}{2 \pi^2} \sigma(T) T \alpha_{em}
\]

Quite different results from previous lattice calculations:

\[
\frac{\sigma}{T} \simeq 7 C_{em}
\]

S. Gupta PLB 597(2004)57

\[
\frac{\sigma}{T} \simeq (0.4 \pm 0.1) C_{em}
\]

G. Aarts et al., PRL 99(2007) 022002

\[N_\tau = 8 - 14, \ T_\sigma \leq 44\]

staggered fermions used

\[N_\tau = 16, 24, \ T_\sigma = 64\]
Spectral function

\[ \rho(\omega, \vec{p}) = D^+ (\omega, \vec{p}) - D^- (\omega, \vec{p}) = 2 \text{Im} D_R (\omega, \vec{p}) \]

Euclidean correlation function

\[ G_{\mu\nu}(\tau, \vec{p}) = \int d^3 x \langle J_\mu (\tau, \vec{x}) J^\dagger_\nu (0, \vec{0}) \rangle e^{i \vec{p} \cdot \vec{x}} \]

\[ J_\mu (\tau, \vec{x}) \equiv \bar{q}(\tau, \vec{x}) \gamma_\mu q(\tau, \vec{x}) \]

Spectral representation

\[ G(\tau, \vec{p}) = \int d^3 x e^{-i \vec{p} \cdot \vec{x}} D^+ (-i \tau, \vec{x}) , \quad D^+(t, \vec{x}) = D^- (t + i \beta, \vec{x}) \]

\[ G_H(\tau, \vec{p}, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho_H (\omega, \vec{p}, T) \frac{\cosh(\omega (\tau - 1/2T))}{\sinh(\omega/2T)} , \quad H = 00, ii, V . \]
Vector correlation function

\begin{equation*}
G_H(\tau, \vec{p}, T) = \int_0^\infty \frac{d\omega}{2\pi} \rho_H(\omega, \vec{p}, T) \frac{\cosh(\omega(\tau - 1/2T))}{\sinh(\omega/2T)} , \quad H = 00, \; ii, \; V .
\end{equation*}

\[ p=0 \text{ in this work} \]

time like correlator \( G_{00} \) and space like correlator \( G_{ii} \)

\[ G_V(\tau, \vec{p}, T) = G_{ii}(\tau, \vec{p}, T) + G_{00}(\tau, \vec{p}, T) \]

conserved current, \( J_0 \), gives \( T \)-independent correlator \( G_{00} \)

\[ G_{00}(T) \equiv -\chi_T T + O(a^2) \]

the local, non-conserved current needs to be renormalized

\[ J_\mu(\tau, \vec{x}) = 2\kappa Z_V \bar{\psi}(\tau, \vec{x})\gamma_\mu \psi(\tau, \vec{x}) \]

avoid ambiguities of renormalization

\[ R(\tau) \equiv \frac{G_V(\tau)}{G_{00}(\tau)} ; \quad R(\tau) \equiv \frac{G_V(\tau)}{G_{00}(\tau)G_V^{free}(\tau T)} \]
Prior information on spectral functions

**free vector spectral function (in the infinite temperature limit)**

\[
\rho_{00}^{\text{free}}(\omega) = -2\pi T^2 \omega \delta(\omega)
\]

\[
\rho_{ii}^{\text{free}}(\omega) = 2\pi T^2 \omega \delta(\omega) + \frac{3}{2\pi} \omega^2 \tanh\left(\frac{\omega}{4T}\right)
\]

*δ*-functions cancel in \(\rho_V(\omega) \equiv \rho_{00}(\omega) + \rho_{ii}(\omega)\)

**vector spectral function at \(T<\infty\)**

*δ*-function in \(\rho_{00}\) is protected

\[
\rho_{00}(\omega, T) = -2\pi \chi_q \omega \delta(\omega)
\]

*δ*-function in \(\rho_{ii}\) is smeared out

possible form: Breit-Wigner (BW) form + modified continuum

\[
\rho_{ii}(\omega, T) = \chi_q c_{BW} \frac{\omega \Gamma}{\omega^2 + (\Gamma/2)^2} + \frac{3}{2\pi} \left(1 + \frac{\alpha_s}{\pi}\right) \omega^2 \tanh\left(\frac{\omega}{4T}\right)
\]

3-4 parameters: \((\chi_q), c_{BW}, \Gamma, \alpha_s\)
Basic ideas of Lattice QCD

Expectation values of the observable $O$ in the path integral formalism

$$\langle O \rangle = \frac{1}{Z} \int D\bar{\psi} D\psi D\bar{\psi} D\psi \, O \, e^{-S_{lat}}$$

$$S_{lat} = S_g + S_f$$

$$Z = \int D\bar{\psi} D\psi D\bar{\psi} D\psi \, e^{-S_{lat}} = \int D\bar{\psi} D\psi \, e^{-S_g} \, \text{det} M_f$$

★ discretize the gauge and fermion fields
★ Monte Carlo: generate an ensemble of the field configuration
★ calculate the observable on every field configuration of the ensemble
★ build ensemble average
★ to save computing time, set $\text{det} M_f = \text{constant}$: quenched approximation
Lattice QCD at finite temperature

- Four dim. Euclidean lattice
  \[ N^3_\sigma \times N_T \]
- Temperature \( T = 1/(N_T a) \)
- \( a \ll \lambda \ll N_\sigma a \)

★ Finite volume effects
★ Lattice cutoff effects

Input parameters
- lattice gauge coupling: \( \beta (= 6/g^2) \)
- lattice size: \( N_T, N_\sigma \)
- quark masses
- .......
Vector correlation functions on large & fine lattices

- SU(3) gauge configurations at $T/T_c \approx 1.45$
- Lattice size $N_\sigma^3 \times N_T$ with $N_\sigma = 32-128$ & $N_T = 16, 24, 32, 48$
- Non-perturbatively clover $O(a)$ improved Wilson fermions
- Quark masses close to chiral limit $\kappa \simeq \kappa_C$

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<th>$N_T$</th>
<th>$N_\sigma$</th>
<th>$\beta$</th>
<th>$c_{sw}$</th>
<th>$\kappa$</th>
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- Volume dependence
- Cut-off dep. & continuum extrapolation
- Close to continuum
Volume & cut-off dep. of vector corr. function

small volume dep.
fix lattice volume $V=128^3$

vary $N_T$ at fixed $T \approx 1.45 T_c$

Normalized by free correlators in the continuum $G_V^{\text{free}}(\tau T)$

$$G_V^{\text{free}}(\tau T) = 6T^3 \left( \pi (1 - 2\tau T) \frac{1 + \cos^2(2\pi \tau T)}{\sin^3(2\pi \tau T)} + 2 \frac{\cos(2\pi \tau T)}{\sin^2(2\pi \tau T)} \right)$$
Volume & cut-off dep. of vector corr. function

Cut-off effects are more severe than finite volume effects.

Large $N_T$ needed to perform continuum extrapolation.

$G_V(\tau T)$ is close to the free case at large $\tau T$.

Incomplete cancelation between $G_{00}(\tau T)$ and BW-contribution to $G_{ii}(\tau T)$?
Cut-off effects of vector corr. function

- ratios are free of renormalization ambiguities
- most of the cut-off effects are well described by discretization errors of non-interacting case
Cut-off effects of vector corr. function

\[ T^2 G_{V}(\tau T)/ [\chi_{q} G_{V}^{\text{free}}(\tau T)] \]

\[ T^2 G_{V}(\tau T)/ [\chi_{q} G_{V}^{\text{free,lat}}(\tau T)] \]

\[ T^2 G_{ii}(\tau T)/ [\chi_{q} G_{ii}^{\text{free}}(\tau T)] \]

\[ T^2 G_{ii}(\tau T)/ [\chi_{q} G_{ii}^{\text{free,lat}}(\tau T)] \]
Continuum extrapolation

• extrapolation in \((aT)^2 = 1/N_T^2\)

<table>
<thead>
<tr>
<th>(N_T)</th>
<th>(\infty)</th>
</tr>
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<tbody>
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<td>(\chi_q/T^2)</td>
<td>0.897(3)</td>
</tr>
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<td>(G_{V(2)}/(\tilde{\chi}<em>q G</em>{V(2)}^{free}))</td>
<td>1.189(13)</td>
</tr>
<tr>
<td>(G_{V(1/2)}/(\tilde{\chi}<em>q G</em>{V(1/2)}^{free}))</td>
<td>1.211(9)</td>
</tr>
<tr>
<td>(G_{V(1/4)}/(\tilde{\chi}<em>q G</em>{V(1/4)}^{free}))</td>
<td>1.190(7)</td>
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<tr>
<td>(G_{ii(1/2)}/(\tilde{\chi}<em>q G</em>{ii(1/2)}^{free}))</td>
<td>1.142(9)</td>
</tr>
<tr>
<td>(G_{ii(1/4)}/(\tilde{\chi}<em>q G</em>{ii(1/4)}^{free}))</td>
<td>1.172(7)</td>
</tr>
</tbody>
</table>

• extrapolation at the other values of \(\tau T\) using spline interpolation on data at fixed cut-off

• extrapolation under control for \(\tau T \geq 0.2\)
• Increase of $G_V(\tau T)/G_V^{\text{free}}(\tau T)$ with $\tau T$ is obvious

• The rise with $\tau T$ indicates that vector spectral function in the low frequency region is different from the free case

• Motivation for the Breit-Wigner type ansatz fitting
Taylor expansion around the mid-point $\tau T = 1/2$

$$G_V(\tau T) = G_V^{(0)} \sum_{n=0}^{\infty} \frac{G_V^{(2n)}}{G_V^{(0)}} \left( \tau T - \frac{1}{2} \right)^{2n} = \int_0^\infty \frac{d\omega}{2\pi} \frac{\rho_V(\omega)}{\sinh(\omega/2T)} \left[ 1 + \frac{1}{2!} \left( \frac{\omega}{T} \right)^2 (\tau T - \frac{1}{2})^2 + \frac{1}{4!} \left( \frac{\omega}{T} \right)^4 (\tau T - \frac{1}{2})^4 + \cdots \right]$$

Thermal moments of vector spectral functions

$$G_V^{(n)} = \frac{1}{n!} \frac{d^n G_V(\tau, T)}{d(\tau T)^n} \bigg|_{\tau T = 1/2} = \frac{1}{n!} \int_0^\infty \frac{d\omega}{2\pi} \left( \frac{\omega}{T} \right)^n \frac{\rho_V(\omega)}{\sinh(\omega/2T)}, \quad G_V^{(0)} = G_V(\tau T = 1/2)$$

Ratio of mid-point subtracted correlation functions

$$\Delta_V(\tau T) = \frac{G_V(\tau T) - G_V^{(0)}}{G_V^{\text{free}}(\tau T) - G_V^{(0), \text{free}}}$$

$$= \frac{G_V^{(2)}}{G_V^{(2), \text{free}}} \left( 1 + \left( \frac{G_V^{(4)}}{G_V^{(2)}} - \frac{G_V^{(4), \text{free}}}{G_V^{(2), \text{free}}} \right) (\tau T - \frac{1}{2})^2 + \cdots \right)$$

$$\frac{G_V^{(2)}}{G_V^{(0)}} = (0.982 \pm 0.012) \frac{G_V^{(2), \text{free}}}{G_V^{(0), \text{free}}}, \quad \frac{G_{ii}^{(2)}}{G_{ii}^{(0)}} = (1.043 \pm 0.010) \frac{G_{ii}^{(2), \text{free}}}{G_{ii}^{(0), \text{free}}}.$$
Taylor expansion around the mid-point $\tau T = 1/2$

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Thermal moments of vector spectral functions

$$G_V^{(n)} = \frac{1}{n!} \left. \frac{d^n G_V(\tau, T)}{d(\tau T)^n} \right|_{\tau T = 1/2} = \frac{1}{n!} \int_0^\infty \frac{d\omega}{2\pi} \left( \frac{\omega}{T} \right)^n \frac{\rho_V(\omega)}{\sinh(\omega/2T)} \quad G_V^{(0)} = G_V(\tau T = 1/2)$$

$$\frac{T^2 G_V(\tau T)}{[\chi_0 G_V^{\text{free}}(\tau T)]}$$

$$\frac{G_V(\tau T)}{G_V^{\text{free}}(\tau T)} = \frac{G_V^{(0)}}{G_V^{(0),\text{free}}} \left( 1 + \left( \frac{G_V^{(2)}}{G_V^{(0)}} - \frac{G_V^{(2),\text{free}}}{G_V^{(0),\text{free}}} \right) \left( \tau T - \frac{1}{2} \right)^2 + \cdots \right)$$

$$\frac{G_V^{(2)}}{G_V^{(0)}} = (0.982 \pm 0.012) \frac{G_V^{(2),\text{free}}}{G_V^{(0),\text{free}}},$$

$$\frac{G_{ii}^{(2)}}{G_{ii}^{(0)}} = (1.043 \pm 0.010) \frac{G_{ii}^{(2),\text{free}}}{G_{ii}^{(0),\text{free}}}.$$
**Breit-Wigner + continuum Ansatz**

- The correlation function at $\tau T=1/2$ is about 2% larger than the corresponding free field value: $G_{ii}(1/2)/G_{ii}^{\text{free}}(1/2) = 1.024(8)$

- The second moment of the vector spectral function deviates from the free field value by about 7%: $G_V^{(2)}/G_V^{(2),\text{free}} = G_{ii}^{(2)}/G_{ii}^{(2),\text{free}} = 1.067(12)$

- The deviation from the free field value increase with increasing Euclidean time: $G_V(1/4)/G_V^{\text{free}}(1/4) = (0.982 \pm 0.005)G_V(1/2)/G_V^{\text{free}}(1/2)$

**Fitting Ansatz**

\[
\begin{align*}
\rho_{00}(\omega, T) &= -2\pi \chi_q \omega \delta(\omega) \\
\rho_{ii}(\omega, T) &= 2\chi_q c_{BW} \frac{\omega \Gamma/2}{\omega^2 + (\Gamma/2)^2} + \frac{3}{2\pi} (1 + k) \omega^2 \tanh\left(\frac{\omega}{4T}\right)
\end{align*}
\]

- smeared delta
- modified free spf
Breit-Wigner + continuum Ansatz

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Fitting Ansatz

$$\frac{\sigma}{T} = \frac{1}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega)}{\omega T} = \frac{C_{em}}{3} \frac{2c_{BW} \tilde{\chi}_q}{\tilde{\Gamma}}$$

$$\rho_{ii}(\omega, T) = 2\chi_q c_{BW} \frac{\omega \Gamma / 2}{\omega^2 + (\Gamma / 2)^2} + \frac{3}{2\pi} (1 + k) \omega^2 \tanh\left(\frac{\omega}{4T}\right)$$

smeared delta  
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- The correlation function at $\tau T = 1/2$ is about 2% larger than the corresponding free field value:
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Fitting Ansatz

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\[ \rho_{ii}(\omega, T) = 2\chi_q c_{BW} \frac{\omega \Gamma/2}{\omega^2 + (\Gamma/2)^2} + \frac{3}{2\pi} (1 + k) \omega^2 \tanh\left(\frac{\omega}{4T}\right) \]

\[ \tilde{\rho}_{ii}(\tilde{\omega}) = \frac{2c_{BW}\tilde{\chi}_q}{\tilde{\Gamma}} \frac{2\tilde{\omega}(\tilde{\Gamma}/2)^2}{\tilde{\omega}^2 + (\tilde{\Gamma}/2)^2} + \frac{3}{2\pi} (1 + k) \tilde{\omega}^2 \tanh\left(\frac{\tilde{\omega}}{4}\right) \]
Fit to correlators & thermal moments

\[ \tilde{\rho}_{ii}(\tilde{\omega}) = \frac{2c_{BW}\tilde{\chi}_q}{\tilde{\Gamma}} \frac{2\tilde{\omega}(\tilde{\Gamma}/2)^2}{\tilde{\omega}^2 + (\tilde{\Gamma}/2)^2} + \frac{3}{2\pi} (1 + k) \tilde{\omega}^2 \tanh\left(\frac{\tilde{\omega}}{4}\right) \]

1. Correlator data extrapolated in the continuum

\[ \tilde{G}_{ii}(\tau T) = c_{BW}\tilde{\chi}_q F_{BW}(\tau T, \tilde{\Gamma}) + (1 + k(T)) \tilde{G}_V^{free}(\tau T) \]

\[ F_{BW}(\tau T, \tilde{\Gamma}) = \frac{\tilde{\Gamma}}{2\pi} \int_0^\infty d\tilde{\omega} \frac{\tilde{\omega}}{(\tilde{\Gamma}/2)^2 + \tilde{\omega}^2} \frac{\cosh(\tilde{\omega}(\tau T - 1/2))}{\sinh(\tilde{\omega}/2)} \]

2. The zeroth and second thermal moments

\[ \tilde{G}_{ii}^{(0)} \equiv \tilde{G}_{ii}(1/2) = c_{BW}\tilde{\chi}_q F_{BW}^{(0)}(\tilde{\Gamma}) + 2 (1 + k(T)) \]

\[ \Delta_V(\tau T) = c_{BW}\tilde{\chi}_q \frac{F_{BW}(\tau T, \tilde{\Gamma}) - F_{BW}(1/2, \tilde{\Gamma})}{G_V^{free}(\tau T) - G_V^{free}(1/2)} + 1 + k(T) \]

\[ \frac{G_V^{(2)}}{G_V^{(2), free}} \equiv \Delta_V(1/2) = c_{BW}\tilde{\chi}_q \frac{F_{BW}^{(2)}(\tilde{\Gamma})}{G_V^{(2), free}} + 1 + k(T) \]

\[ k = \alpha_s / \pi \]

\[ g^2(T) = 4\pi\alpha_s \simeq 1.6 \]

\[ k = 0.0465(30), \quad \tilde{\Gamma} = 2.235(75), \quad 2c_{BW}\tilde{\chi}_q/\tilde{\Gamma} = 1.098(27) \]

\[ \chi^2/d.o.f. = 0.06, \quad d.o.f. = 12 \]

good agreement with Kaczmarek et al., PRD70(2004)074505
Dependences on the width

\[ \tilde{\rho}_{ii}(\tilde{\omega}) = \frac{2c_{BW}\tilde{\chi}_q}{\tilde{\Gamma}} \frac{2\tilde{\omega}(\tilde{\Gamma}/2)^2}{\tilde{\omega}^2 + (\tilde{\Gamma}/2)^2} + \frac{3}{2\pi} (1 + k) \tilde{\omega}^2 \tanh\left(\frac{\tilde{\omega}}{4}\right) \]

\[ k = 0.0465(30), \quad \tilde{\Gamma} = 2.235(75), \quad 2c_{BW}\tilde{\chi}_q/\tilde{\Gamma} = 1.098(27) \]

- vector correlation function is sensitive to the low energy, Breit-Wigner contribution only for distance \( \tau T \gtrsim 0.25 \)

- fit parameters can be well constrained by the additional second thermal moment

\( \tilde{T}^2G_{V}(\tau T)/(\chi_qG_{V}^{\text{free}}(\tau T)) \)

\( 2Tc_{BW}/\Gamma=1.098 \)

\( \Gamma/2T=1.080 \)

\( 1.117 \) - black line

\( 1.155 \) - red line

\( \tau T \)
Estimate of electrical conductivity

\[
\tilde{\rho}_{ii}(\tilde{\omega}) = \frac{2c_{BW}\tilde{\chi}_q}{\tilde{\Gamma}} \frac{2\tilde{\omega}(\tilde{\Gamma}/2)^2}{\tilde{\omega}^2 + (\tilde{\Gamma}/2)^2} + \frac{3}{2\pi} (1 + k) \tilde{\omega}^2 \tanh\left(\frac{\tilde{\omega}}{4}\right)
\]

\[k = 0.0465(30)\quad \tilde{\Gamma} = 2.235(75)\quad 2c_{BW}\tilde{\chi}_q/\tilde{\Gamma} = 1.098(27)\]

\[
\frac{\rho_{ii}(\omega)}{\omega T} = \frac{C_{em}}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega)}{\omega T} = \frac{C_{em}}{3} \frac{2c_{BW}\tilde{\chi}_q}{\tilde{\Gamma}} = (0.37 \pm 0.01)C_{em}
\]
Estimate of electrical conductivity

\[ \tilde{\rho}_{ii}(\tilde{\omega}) = \frac{2c_{BW} \tilde{\chi}_q}{\tilde{\Gamma}} \frac{2\tilde{\omega}(\tilde{\Gamma}/2)^2}{\tilde{\omega}^2 + (\tilde{\Gamma}/2)^2} + \frac{3}{2\pi} (1 + k) \tilde{\omega}^2 \tanh\left(\frac{\tilde{\omega}}{4}\right) \]

\[ k = 0.0465(30), \quad \tilde{\Gamma} = 2.235(75), \quad 2c_{BW} \tilde{\chi}_q / \tilde{\Gamma} = 1.098(27) \]

\[ \frac{\sigma}{T} = \frac{C_{em}}{6} \lim_{\omega \to 0} \frac{\rho_{ii}(\omega)}{\omega T} \]

\[ = \frac{C_{em}}{3} \frac{2c_{BW} \tilde{\chi}_q}{\tilde{\Gamma}} \]

\[ = (0.37 \pm 0.01) C_{em} \]

(accidentally) close to Aarts’ result!
Breit-Wigner + truncated continuum Ansatz

\[ \rho_{ii}(\omega) = 2\chi_q c_{bw} \frac{\omega \Gamma / 2}{\omega^2 + (\Gamma / 2)^2} + \frac{3}{2\pi} \left( 1 + k \right) \omega^2 \tanh\left( \frac{\omega}{4T} \right) \Theta(\omega_0, \Delta_\omega) \]

\[ \Theta(\omega_0, \Delta_\omega) = \left( 1 + e^{(\omega_0^2 - \omega^2) / \omega \Delta_\omega} \right)^{-1} \]

delay the onset \((\omega_0)\) of the continuum part

- Rise of BW peaks compensate for the cut from continuum parts
- Fits become worse with increasing \(\omega_0\) and/or increasing \(\Delta_\omega\)
Breit-Wigner + truncated continuum Ansatz

\[
\rho_{ii}(\omega) = 2\chi_q c_{BW} \frac{\omega \Gamma/2}{\omega^2 + (\Gamma/2)^2} + \frac{3}{2\pi} \frac{(1 + k)}{2} \omega^2 \tanh\left(\frac{\omega}{4T}\right) \Theta(\omega_0, \Delta_{\omega})
\]

\[
\Theta(\omega_0, \Delta_{\omega}) = \left(1 + e^{(\omega_0^2 - \omega^2)/\omega_0 \Delta_{\omega}}\right)^{-1}
\]

delay the onset ($\omega_0$) of the continuum part

\[-\frac{\omega^2}{4T} / \Theta(\omega, \Delta_{\omega})
\]

\[-\frac{\omega^2}{4T} / \Theta(\omega_0, \Delta_{\omega} = \left(1 + e^{(\omega_0^2 - \omega^2)/\omega_0 \Delta_{\omega}}\right)^{-1}
\]

- Ratio of thermal moments reacts sensitive to the truncation of the cont. part
- Spectral function should not deviate much from free behavior for $\omega / T \gtrsim (2 - 4)$
• Extract spectral function (spf) without ansatz

\[ G(\tau, T) = \int_0^\infty \frac{d\omega}{2\pi} \frac{\cosh(\omega(\tau - 1/2T))}{\sinh(\omega/2T)} \rho(\omega, T) \]

\( \mathcal{O}(10) \) Discretized  \( \mathcal{O}(10^3) \) Continuous

• Maximum Entropy Method (MEM)
  • successful in condensed matter physics, astrophysics, image processing...
  • A method to obtain the most probable image from insufficient data

\( \chi^2 \) fitting inconclusive
Based on the Bayesian theorem

\[ P[X|Y] = \frac{P[Y|X]P[X]}{P[Y]} \], \quad P[X|Y]: \text{Probability of } X \text{ given } Y

Ingredients of MEM

\[ P[\rho|GH] \propto P[G|\rho H] P[\rho|H] \]

\[ P[G|\rho H] \propto \exp(-\chi^2/2) \]: likelihood function

\[ P[\rho|H] \propto \exp(\alpha S) \]: prior probability

Shannon-Jaynes entropy:

\[ S = \int_0^\infty d\omega \left[ \rho(\omega) - m(\omega) - \rho(\omega) \ln \left( \frac{\rho(\omega)}{m(\omega)} \right) \right] \]

Default Model (DM): \( m(\omega) \), includes the prior information of \( \rho \), e.g. \( \rho \) is positive-definite

DM is the only input parameter in the MEM algorithm

Maximizing \( P[\rho|GH] \) gives the most probable image
- Differences are smaller than 2% with smaller parameters and increase with worse default models
MEM analysis

- Differences are smaller than 2% with smaller parameters and increase with worse default models
- MEM analysis reproduces the ratio of thermal moments even better
- Statistical errors are small
Thermal dilepton rate & electrical conductivity

\[ \rho_{ii}(\omega)/\omega T \]

\[ \omega_0/T=0, \Delta\omega_0/T=0 \]
\[ \omega_0/T=1.5, \Delta\omega_0/T=0.5 \]

HTL, free

\[ \omega/T \]
Thermal dilepton rate & electrical conductivity

\[ \frac{dN_{l+1}}{d\omega d^3p} = C_{em} \frac{\alpha_{em}^2 \rho_V(\omega, \vec{p}, T)}{6\pi^3 (\omega^2 - \vec{p}^2)(e^{\omega/T} - 1)} \]

**Graphs:**
- **Left Graph:**
  - \( \rho_{ii}(\omega)/\omega T \)
  - \( \omega_0/T=0, \Delta_\omega/T=0 \)
  - \( \omega_0/T=1.5, \Delta_\omega/T=0.5 \)
  - HTL
  - free

- **Right Graph:**
  - \( dN_{l+1}/d\omega \ d^3p \)
  - p=0
  - BW+continuum: \( \omega_0/T=0, \Delta_\omega/T=0 \)
  - \( \omega_0/T=1.5, \Delta_\omega/T=0.5 \)
  - HTL
  - Born
Thermal dilepton rate & electrical conductivity

\[ \frac{dN_{\ell^+\ell^-}}{d\omega d^3p} = C_{em} \frac{\sigma_{em}^2}{6\pi^3} \frac{\rho_V(\omega, \vec{p}, T)}{(\omega^2 - \vec{p}^2)(e^{\omega/T} - 1)} \]

1/3 \approx \frac{1}{C_{em} \frac{\sigma}{T}} \approx 1 \text{ at } T \approx 1.45 \, T_c

\[ \lim_{\omega \to 0} \omega \frac{dR_\gamma}{d^3p} = (0.0004 - 0.0013) T_c^2 \simeq (1 - 3) \cdot 10^{-5} \, \text{GeV}^2 \text{ at } T \approx 1.45 \, T_c \]

\[ \rho_{ii}(\omega)/\omega T \]

\[ \omega/0, \Delta_\omega/0 \]

\[ \omega/0, \Delta_\omega/0.5 \]

HTL

free

\[ \omega/0, \Delta_\omega/1.5, \Delta_\omega/0.5 \]

\[ \omega/0, \Delta_\omega/1.5, \Delta_\omega/0.5 \]

HTL

Born
• We calculated the vector correlation function at $T \approx 1.45T_c$ in quenched lattice QCD and performed a continuum extrapolation.

• $G_V(\tau T)$ is well reproduced using a Breit-Wigner plus continuum ansatz for the vector spectral function.

• Electrical conductivity $\frac{1}{3} \lesssim \frac{1}{C_{em} \frac{\sigma}{T}} \lesssim 1$ at $T \approx 1.45 T_c$.

• Dilepton rate approaches leading order Born rate at $\omega/T \approx 4$. 